OAK RIDGE National Laboratory

July 13-15, 2026, Colorado State University

Hall effects and correlations in quantum materials

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Materials Science and Technology Division

Outline

Various Hall effects
- Ordinary Hall effect
- Integer quantum Hall effect
- Fractional quantum Hall effect
- Anomalous Hall effect
- Spin Hall effect

Ordinary Hall effect
- Semiclassical description
- Example of material specifics

Day 1

Anomalous Hall effect
- Berry curvature in momentum space
- Berry curvature in real space:
  topological Hall effect
- Example

Day 2

Spin Hall effect
- Intrinsic mechanisms
- Extrinsic mechanisms
- Phonon mechanisms
- Spin fluctuation mechanisms

Chern insulators
- Integer Chern insulators
- Fractional Chern insulators

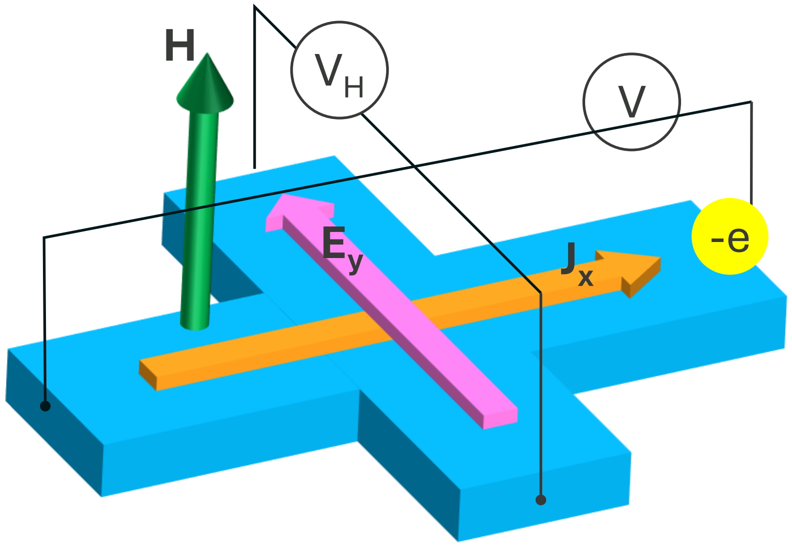
Day 3

Ordinary Hall effect

On a New Action of the Magnet on Electric Currents.

BY E. H. HALL, *Fellow of the Johns Hopkins University.*

American Journal of Mathematics **2**, 287 (1879).



$$R_H = \frac{E_y}{J_x H_z}$$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$R_H = \frac{\sigma_{xy}}{\sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}} \frac{1}{H_z}$$

$$R_H = \frac{1}{nqc}$$

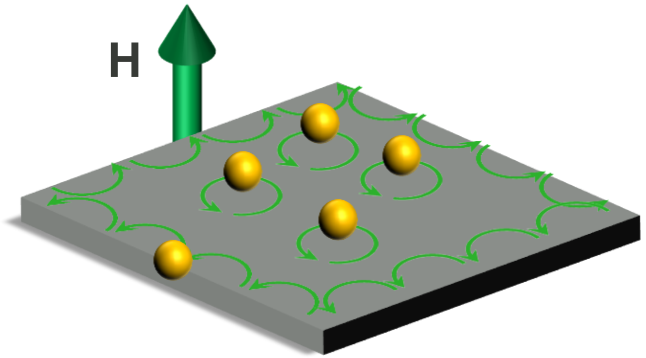
Hall effect was discovered more than a century ago.
The US was 103 years old.

$$\mathbf{F} = q_e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right)$$

Integer quantum Hall effect: Experiment

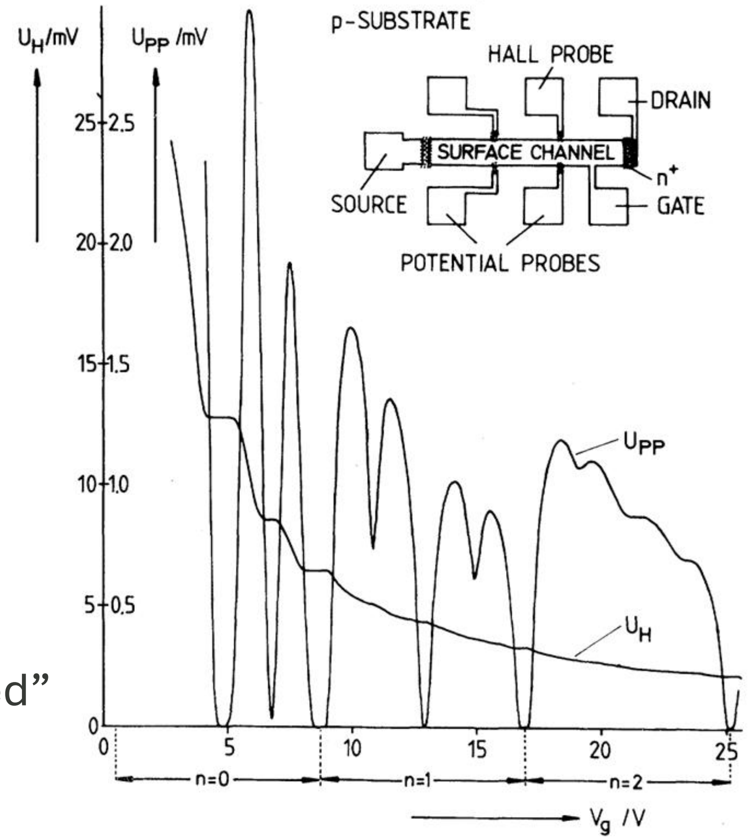
K. v. Klitzing, G. Dorda, and M. Pepper, *New Method for High-Accuracy Determination of the Fine-Structure Constant Based on Quantized Hall Resistance*, Phys. Rev. Lett. **45**, 494 (1980).

J.-I. Wakabayashi and S. Kawaji, *Hall Effect in Silicon MOS Inversion Layers under Strong Magnetic Fields*, J. Phys. Soc. Jpn. **44**, 1839 (1978).



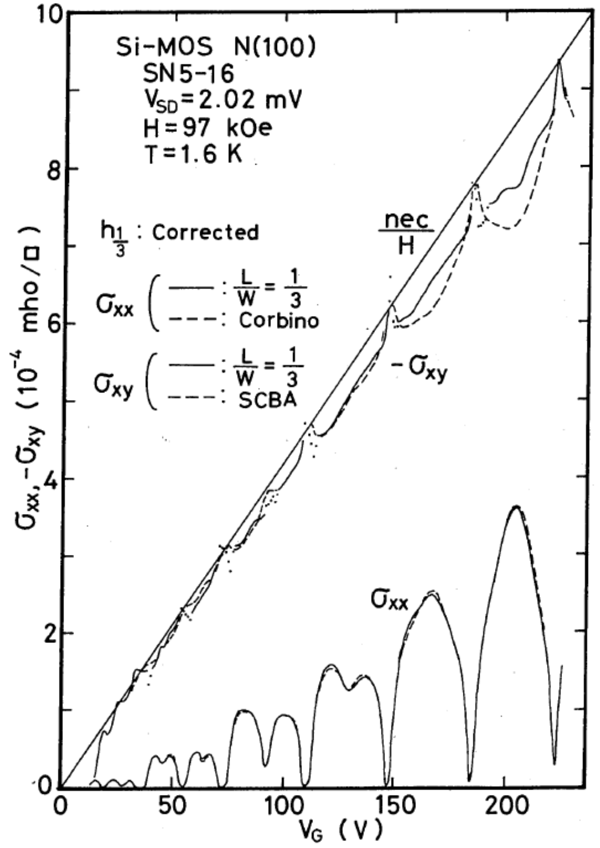
Integer quantum Hall effect was “discovered” a century after the discovery of Hall effect.

Voltage



v. Klitzing et al. Phys. Rev. Lett. **45**, 494 (1980).

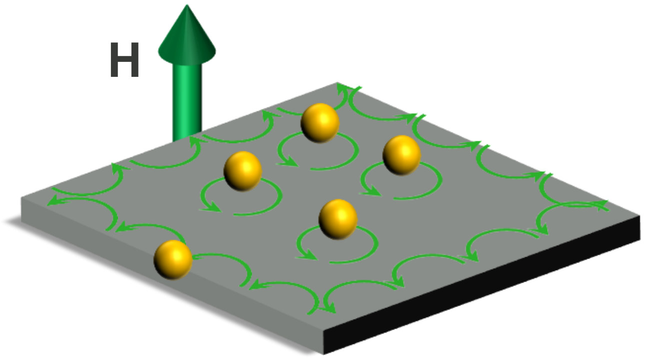
Conductivity



Wakabayashi and Kawaji, J. Phys. Soc. Jpn. **44**, 1839 (1978).

Integer quantum Hall effect: Theory

T. Ando, Y. Matsumoto, and Y. Uemura, J. Phys. Soc. Jpn. **39**, 279 (1975).
D. J. Thouless, M. Kohmoto, M. P. Nightingale, and M. den Nijs, Phys. Rev. Lett. **49**, 405 (1982).



AMU theory

$$\sigma_{xy} = -\frac{nec}{H} + \Delta\sigma_{xy}$$

$$\Delta\sigma_{XY}^{(2)} = \frac{e^2}{4\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right)\mathrm{d}E \sum_{N_1}\sum_{\pm}(\pm i)\xi^{x}_{N_1 N_1\pm 1}(E+i0, E-i0)$$
$$\times G_{N_1}(E+i0)G_{N_1\pm1}(E-i0)\Xi^{x}_{N_1\pm1 N_1}(E-i0, E+i0)\ .$$

$$\sigma_{xy} \to -\frac{nec}{H} \qquad n = e(N+1)H/2\pi\hbar c$$

TKNN theory

$$\sigma_{\mathrm{H}} = \frac{ie^2}{A\hbar}\sum_{\epsilon_\alpha < E_F}\sum_{\epsilon_\beta > E_F}\frac{(\partial\hat{H}/\partial k_1)_{\alpha\beta}(\partial\hat{H}/\partial k_2)_{\beta\alpha} - (\partial\hat{H}/\partial k_2)_{\alpha\beta}(\partial\hat{H}/\partial k_1)_{\beta\alpha}}{(\epsilon_\alpha - \epsilon_\beta)^2}$$

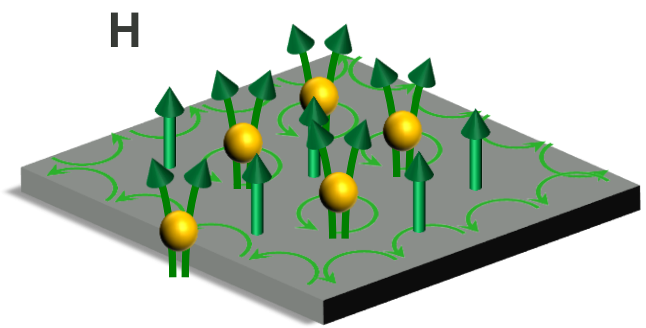
$$\sigma_{\mathrm{H}} = \frac{ie^2}{2\pi h}\sum\int d^2k\int d^2r\left(\frac{\partial u^*}{\partial k_1}\frac{\partial u}{\partial k_2} - \frac{\partial u^*}{\partial k_2}\frac{\partial u}{\partial k_1}\right)$$
$$= \frac{ie^2}{4\pi h}\sum\oint dk_j\int d^2r\left(u^*\frac{\partial u}{\partial k_j} - \frac{\partial u^*}{\partial k_j}u\right),$$

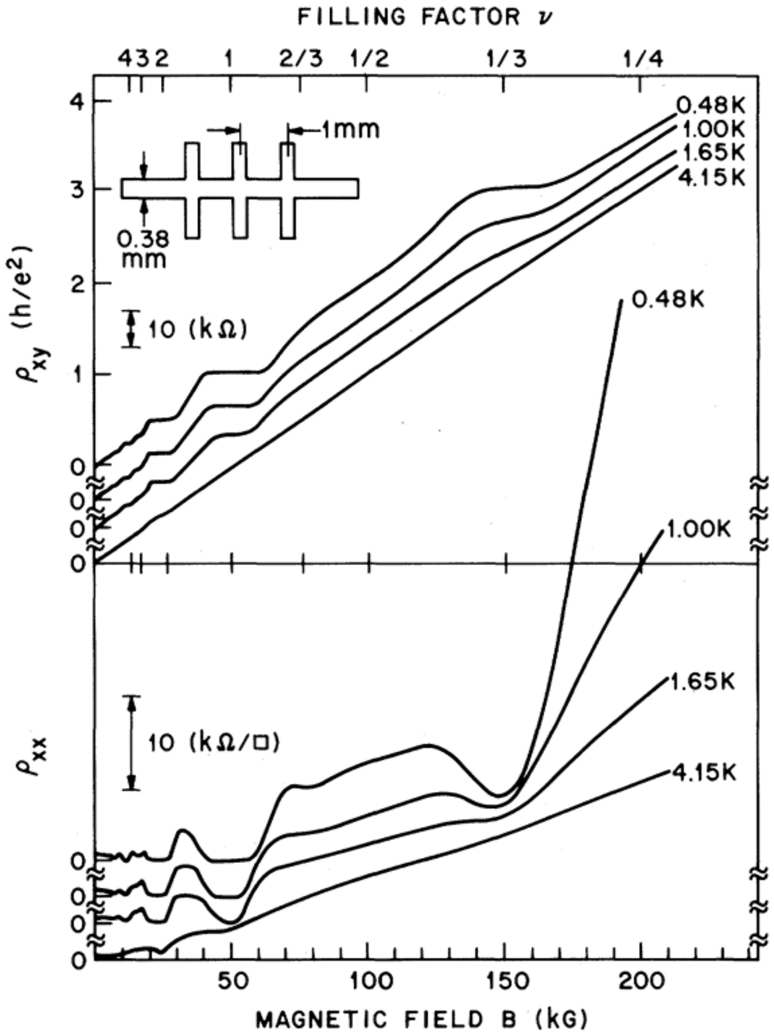
Integer quantum Hall effect was theoretically "predicted" before the experimental confirmation.

Fractional quantum Hall effect

D. C. Tsui, H. L. Stormer, and A. C. Gossard, Phys. Rev. Lett. **48**, 1559 (1982).
R. B. Laughlin, Phys. Rev. Lett. **50**, 1395 (1983).
Q. Niu, D.J. Thouless, and Y.-S. Wu, Phys. Rev. B **31**, 3372 (1985).

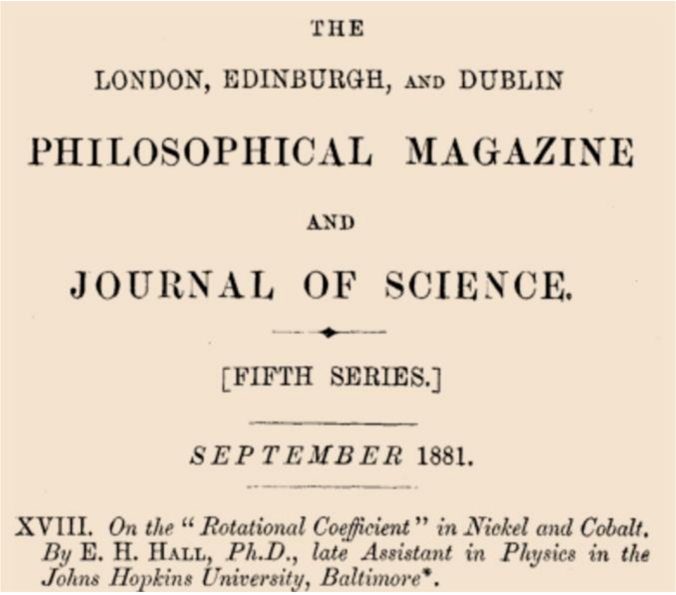
$$\sigma_{xy} = -\frac{e^2}{h}\nu \qquad \nu = nh/eB$$





Fractional quantum Hall effect is a new phenomena due to many-body effects.

Anomalous Hall effect


THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

SEPTEMBER 1881.

XVIII. *On the "Rotational Coefficient" in Nickel and Cobalt.* *By* E. H. HALL, *Ph.D., late Assistant in Physics in the Johns Hopkins University, Baltimore**.

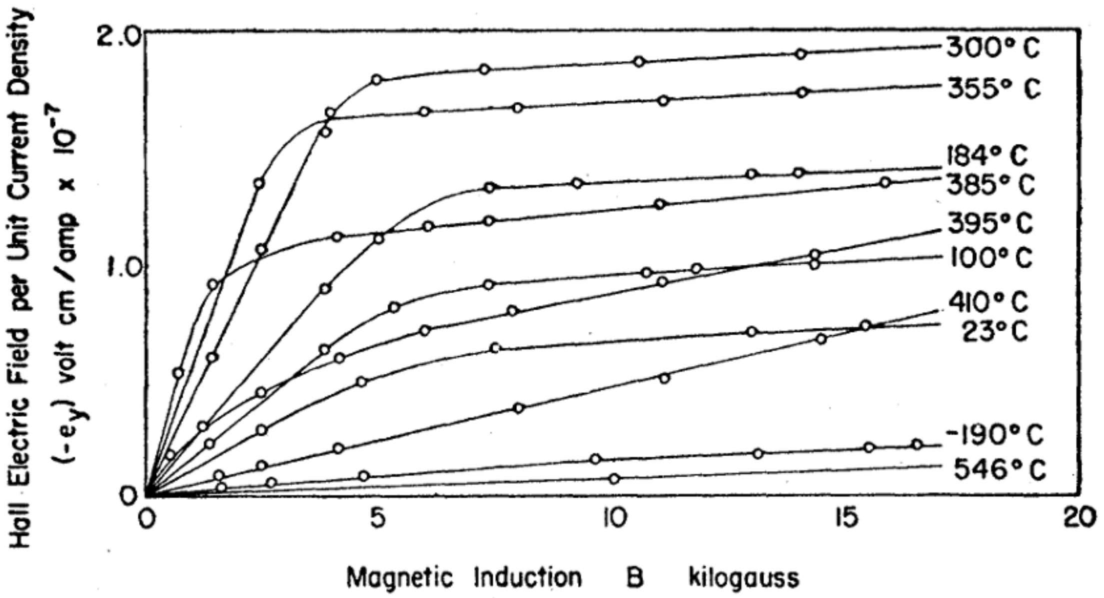


FIG. 1. Hall effect in nickel from the data of A. W. Smith.

A. W. Smith, Phys. Rev. **30**,1 (1910).
E. M. Pugh and N. Rostoker, Rev. Mod. Phys. **25**, 151 (1953).

$$E_y = R_0 H_z + R_1 M_z$$

Anomalous Hall effect was discovered by Hall himself only two years after the discovery of ordinary Hall effect. It took more than half a century to understand its mechanism (may not be fully yet).

Anomalous Hall effect: Various mechanisms

Intrinsic mechanism
R. Karplus and J. M. Luttinger, Phys. Rev. **95**, 1154 (1954).
F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

Skew scattering mechanism
J. Smit, Physica (Amsterdam) **21**, 877 (1955); 24, 39 (1958).

Side-jump mechanism
L. Berger, Phys. Rev. B **2**, 4559 (1970); **5**, 1862 (1972).

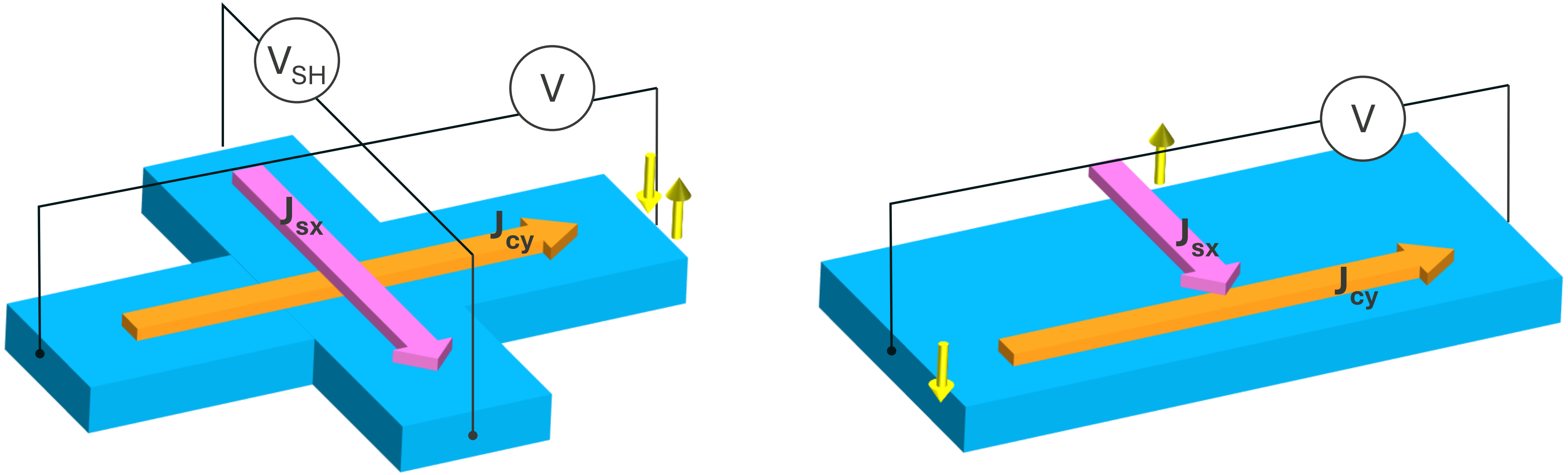
Skew scattering and side jump
A. Crépieux and P. Bruno, Phys. Rev. B **64**, 014416 (2001).

Intrinsic-skew scattering crossover
S. Onoda, N. Sugimoto, and N. Nagaosa, Phys. Rev. Lett. **97**, 126602 (2006).

Spin Hall effect

M.I. Dyakonov and V.I. Perel, JETP Lett. **13**, 467 (1971); Phys. Lett. **35A**, 459 (1971).
J. E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).

Inverse spin-Hall effect
E. Saitoh, M. Ueda, H. Miyajima, and G. Tatara, Appl. Phys. Lett. **88**, 182509 (2006).



Spin Hall effect could be useful and important for next generation electronics, spintronics and quantum applications, because it allows the manipulation of "spin" without using a magnetic field.

L. Zhu, D. C. Ralph, and R. A. Buhrman, Phys. Rev. Applied **10**, 031001 (2018). $\sigma^{sH} \sim 7.5 \times 10^5\ \Omega^{-1}\mathrm{m}^{-1}$ in Au-Pt alloys
C.-Y. Hu et al. ACS Appl. Electron. Mater. **4**,1099 (2022). $\sigma^{sH} \sim 6.5 \times 10^5\ (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ in Pt-Cr alloys
S. Rho et al. ACS Nano **19**, 38749 (2025). $\sigma^{sH} \sim 7.5 \times 10^5\ \Omega^{-1}\mathrm{m}^{-1}$ in topological semimetal $Bi_{1-x}Sb_x$
G. Ji et al. Nature Materials (2026). $\sigma^{sH} \sim 1.6 \times 10^7 (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ $PtCoO_2$

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Spin Hall effect: Various mechanisms

Intrinsic mechanism

S. Murakami, N. Nagaosa, and S.-C. Zhang, Science **301**, 5638 (2003); Phys. Rev. Lett. **93**, 156804 (2004).
C. L. Kane and E. J. Mele, Phys. Rev. Lett. **95**, 146802; 226801 (2005).
B. A. Bernevig and S.-C. Zhang, Phys. Rev. Lett. **96**, 106802 (2006).

⇒ **Topological insulators**

J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. **92**, 126603 (2004).
G. Y. Guo, Y. Yao, and Q. Niu, Phys. Rev. Lett. **94**, 226601 (2005).
G. Y. Guo, S. Murakami, T.-W. Chen, and N. Nagaosa, Phys. Rev. Lett. **100**, 096401 (2008).
L. Wang, R. J. H. Wesselink, Y. Liu, Z. Yuan, K. Xia, and P. J. Kelly, Phys. Rev. Lett. **116**, 196602 (2016).

Skew scattering and side-jump mechanism

W.-K. Tse and S. Das Sarma, Phys. Rev. Lett. **96**, 056601 (2006).

Phonon

C. Gorini, U. Eckern, and R. Raimondi, Phys. Rev. Lett. **115**, 076602 (2015).
C. Xiao, Y. Liu, Z. Yuan, S. A. Yang, and Q. Niu, Phys. Rev. B **100**, 085425 (2019).

Spin fluctuation

D.H. Wei, Y. Niimi, B. Gu, T. Ziman, S. Maekawa, and Y. Otani, Nat. Commun. **3**, 1058 (2012).
B. Gu, T. Ziman, and S. Maekawa, Phys. Rev. B **86**, 241303(R) (2012).
S. Okamoto, T. Egami, and N. Nagaosa, Phys. Rev. Lett. **123**, 196603 (2019).
S. Okamoto and N. Nagaosa, npj Quantum Mater. **9**, 29 (2024).

Ordinary Hall effect

Ordinary Hall effect

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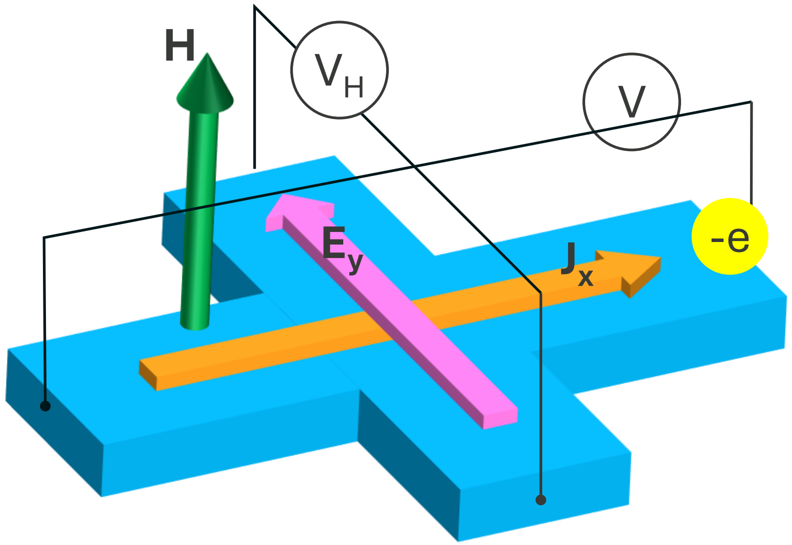

$$R_H = \frac{E_y}{J_x H_z}$$

$$\begin{pmatrix} J_x \\ J_y \end{pmatrix} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{yx} & \sigma_{yy} \end{pmatrix} \begin{pmatrix} E_x \\ E_y \end{pmatrix}$$

$$R_H = \frac{\sigma_{xy}}{\sigma_{xx}\sigma_{yy} - \sigma_{xy}\sigma_{yx}} \frac{1}{H_z}$$

$$R_H = \frac{1}{nqc}$$

Hall effect was discovered more than a century ago.
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$$\mathbf{F} = q_e \left(\mathbf{E} + \frac{1}{c} \mathbf{v} \times \mathbf{H} \right)$$

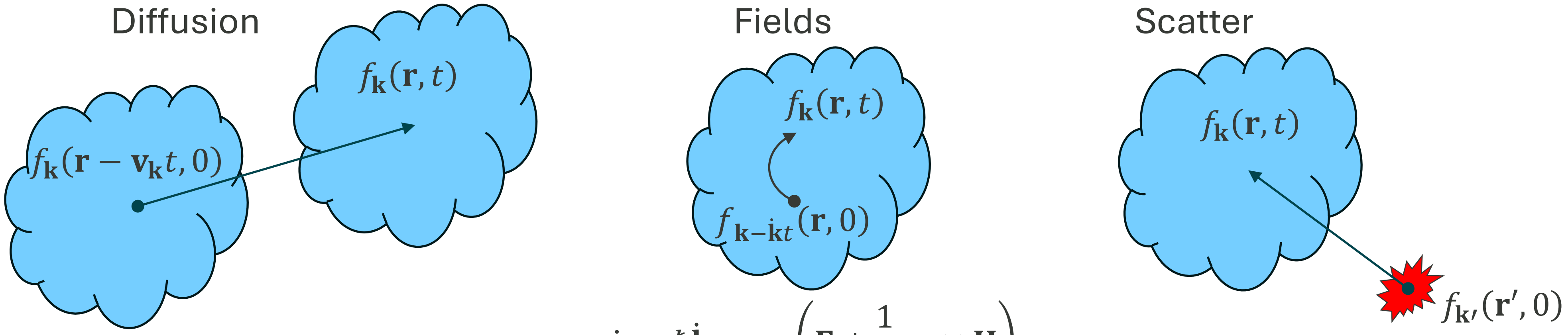
Semiclassical description of ordinary Hall effect

Boltzmann equation approach

Distribution function of particles with momentum $\mathbf{k}$ at position $\mathbf{r}$ and time t: $f_{\mathbf{k}}(\mathbf{r},t)$

Observable: $\langle \mathcal{O} \rangle = \int dk\, dr\, dt\, f_{\mathbf{k}}(\mathbf{r},t)\mathcal{O}(\mathbf{k},\mathbf{r},t)$

Physical processes contribution to $f_{\mathbf{k}}(\mathbf{r},t)$:



$$\dot{\mathbf{p}} = \hbar\dot{\mathbf{k}} = q_e\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_{\mathbf{k}} \times \mathbf{H}\right)$$

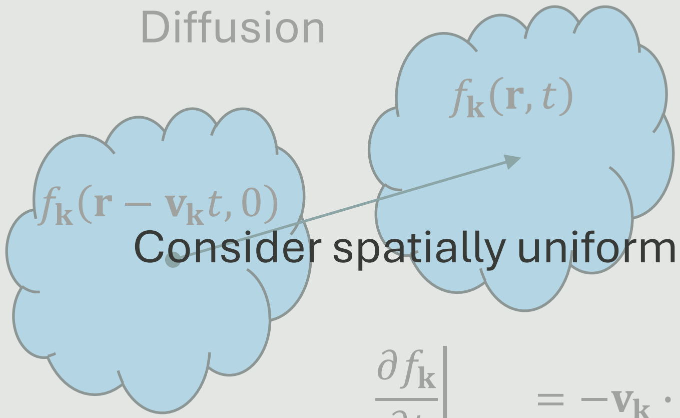
Differential equation for $f_{\mathbf{k}}(\mathbf{r},t)$ in a steady state:

$$\boxed{\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{diff} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{field} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{scatt} = 0}$$

Semiclassical description of ordinary Hall effect

Boltzmann equation approach

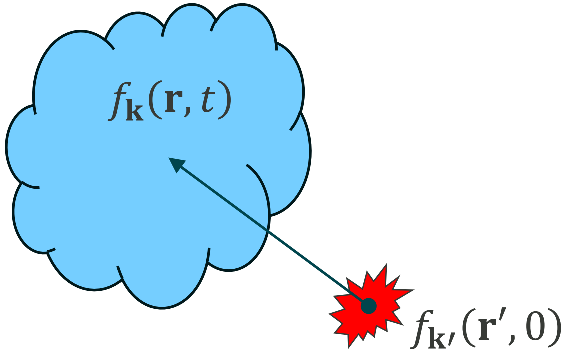
Diffusion



Consider spatially uniform situations

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{diff} = -\mathbf{v}_{\mathbf{k}} \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{r}} = -\mathbf{v}_{\mathbf{k}} \cdot \nabla f_{\mathbf{k}}$$

Scatter



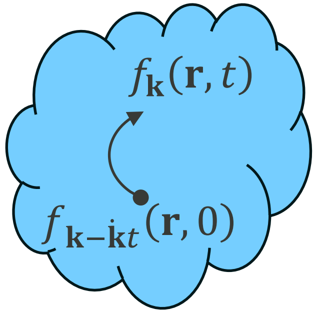
$$f_{\mathbf{k}}(\mathbf{r},t) = f_{\mathbf{k}}^0 + g_{\mathbf{k}}(\mathbf{r},t)$$

Relaxation-time approximation

$$g_{\mathbf{k}}(\mathbf{r},t) = g_{\mathbf{k}}(\mathbf{r},0)e^{-t/\tau_{\mathbf{k}}}$$

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{scatt} = -\frac{1}{\tau_{\mathbf{k}}} g_{\mathbf{k}}$$

Fields



$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{field} = -\dot{\mathbf{k}} \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = -\frac{q_e}{\hbar}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_{\mathbf{k}} \times \mathbf{H}\right) \cdot \frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$\frac{\partial f_{\mathbf{k}}}{\partial \mathbf{k}} = \frac{\partial \varepsilon_{\mathbf{k}}}{\partial \mathbf{k}}\frac{\partial f_{\mathbf{k}}^0}{\partial \varepsilon_{\mathbf{k}}} + \frac{\partial g_{\mathbf{k}}}{\partial \mathbf{k}} = \hbar \mathbf{v}_{\mathbf{k}} \frac{\partial f_{\mathbf{k}}^0}{\partial \varepsilon_{\mathbf{k}}} + \frac{\partial g_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$\left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{field} + \left.\frac{\partial f_{\mathbf{k}}}{\partial t}\right|_{scatt} = 0 \quad \Rightarrow \quad \boxed{-\frac{q_e}{\hbar}\left(\mathbf{E} + \frac{1}{c}\mathbf{v}_{\mathbf{k}} \times \mathbf{H}\right) \cdot \left(\hbar \mathbf{v}_{\mathbf{k}} \frac{\partial f_{\mathbf{k}}^0}{\partial \varepsilon_{\mathbf{k}}} + \frac{\partial g_{\mathbf{k}}}{\partial \mathbf{k}}\right) - \frac{1}{\tau_{\mathbf{k}}} g_{\mathbf{k}} = 0}$$

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Semiclassical description of ordinary Hall effect

Boltzmann equation approach

$$-\frac{q_e}{\hbar}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\mathbf{k}\times\mathbf{H}\right)\cdot\left(\hbar\mathbf{v}_\mathbf{k}\frac{\partial f_\mathbf{k}^0}{\partial\varepsilon_\mathbf{k}}+\frac{\partial g_\mathbf{k}}{\partial\mathbf{k}}\right)-\frac{1}{\tau_\mathbf{k}}g_\mathbf{k}=0 \qquad g_\mathbf{k}=-\frac{q_e\tau_\mathbf{k}}{\hbar}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\mathbf{k}\times\mathbf{B}\right)\cdot\left(\hbar\mathbf{v}_\mathbf{k}\frac{\partial f_\mathbf{k}^0}{\partial\varepsilon_\mathbf{k}}+\frac{\partial g_\mathbf{k}}{\partial\mathbf{k}}\right)$$

Expand $g_\mathbf{k}$ according to the order of external fields:

$$g_\mathbf{k}=g_\mathbf{k}^{(1)}+g_\mathbf{k}^{(2)}+\cdots \qquad \begin{gathered} g_\mathbf{k}^{(1)}\propto E \\ g_\mathbf{k}^{(2)}\propto EH \\ \vdots \end{gathered}$$

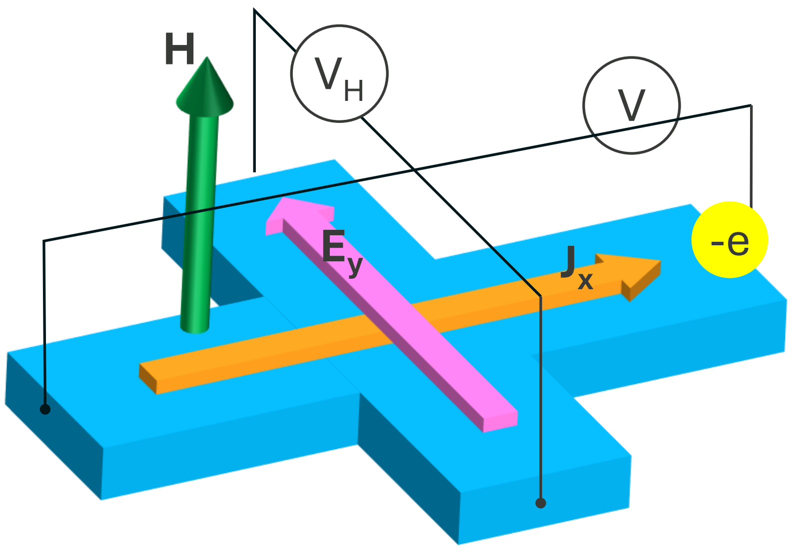
$$g_\mathbf{k}^{(1)}+g_\mathbf{k}^{(2)}+\cdots=-\frac{q_e\tau_\mathbf{k}}{\hbar}\left(\mathbf{E}+\frac{1}{c}\mathbf{v}_\mathbf{k}\times\mathbf{H}\right)\cdot\left(\hbar\mathbf{v}_\mathbf{k}\frac{\partial f_\mathbf{k}^0}{\partial\varepsilon_\mathbf{k}}+\frac{\partial\left\{g_\mathbf{k}^{(1)}+g_\mathbf{k}^{(1)}+\cdots\right\}}{\partial\mathbf{k}}\right)$$

$$g_\mathbf{k}^{(1)}=\frac{q_e\tau_\mathbf{k}}{\hbar}\mathbf{E}\cdot\hbar\mathbf{v}_\mathbf{k}\left(-\frac{\partial f_\mathbf{k}^0}{\partial\varepsilon_\mathbf{k}}\right)$$

$$g_\mathbf{k}^{(2)}=-\frac{q_e\tau_\mathbf{k}}{\hbar}\left(\frac{1}{c}\mathbf{v}_\mathbf{k}\times\mathbf{H}\right)\cdot\frac{\partial g_\mathbf{k}^{(1)}}{\partial\mathbf{k}}=\frac{q_e\tau_\mathbf{k}}{c\hbar}(\mathbf{H}\times\mathbf{v}_\mathbf{k})\cdot\frac{\partial g_\mathbf{k}^{(1)}}{\partial\mathbf{k}}$$

Contribution to the ordinary Hall conductivity $\quad \mathbf{J}=\frac{1}{(2\pi)^3}\int d^3k\, q_e\mathbf{v}_\mathbf{k}\, g_\mathbf{k}^{(2)}$

Semiclassical description of ordinary Hall effect



$$\mathbf{J} = (J_x, 0,0)$$
$$\mathbf{E} = (0, E_y, 0)$$
$$\mathbf{H} = (0,0, H_z)$$

$$g_{\mathbf{k}}^{(1)} = \frac{q_e \tau_{\mathbf{k}}}{\hbar} \mathbf{E} \cdot \hbar \mathbf{v}_{\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right) \rightarrow q_e \tau_{\mathbf{k}} E_y v_{y\mathbf{k}} \left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right)$$

$$g_{\mathbf{k}}^{(2)} = \frac{q_e \tau_{\mathbf{k}}}{c\hbar} (\mathbf{H} \times \mathbf{v}_{\mathbf{k}}) \cdot \frac{\partial g_{\mathbf{k}}^{(1)}}{\partial \mathbf{k}} \rightarrow \frac{q_e \tau_{\mathbf{k}}}{c\hbar} (-H_z v_{y\mathbf{k}}, H_z v_{x\mathbf{k}}, 0) \cdot \frac{\partial g_{\mathbf{k}}^{(1)}}{\partial \mathbf{k}}$$

$$J_x = \frac{1}{(2\pi)^3} \int d^3k\, q_e v_{x\mathbf{k}}\, g_{\mathbf{k}}^{(2)} \qquad \mathbf{v}_{\mathbf{k}} = \frac{1}{\hbar}\frac{\partial \varepsilon_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$= \frac{1}{(2\pi)^3} \int d^3k \frac{q_e^3 \tau_{\mathbf{k}}^2}{c\hbar} v_{x\mathbf{k}} \left\{-v_{y\mathbf{k}} \frac{\partial v_{y\mathbf{k}}}{\partial k_x} + v_{x\mathbf{k}} \frac{\partial v_{y\mathbf{k}}}{\partial k_y}\right\} \left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right) E_y H_z$$

$$= \frac{1}{(2\pi)^3} \int d^3k \frac{q_e^3 \tau_{\mathbf{k}}^2}{c\hbar^4} \left\{\left(\frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_x}\right)^2 \frac{\partial^2 \varepsilon_{\mathbf{k}}}{\partial k_y^2} - \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_x}\frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_y}\frac{\partial^2 \varepsilon_{\mathbf{k}}}{\partial k_x \partial k_y}\right\} \left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right) E_y H_z$$

$$\boxed{\sigma_{xy} = \frac{1}{(2\pi)^3} \int d^3k \frac{q_e^3 \tau_{\mathbf{k}}^2}{c\hbar^4} \left\{\left(\frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_x}\right)^2 \frac{\partial^2 \varepsilon_{\mathbf{k}}}{\partial k_y^2} - \frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_x}\frac{\partial \varepsilon_{\mathbf{k}}}{\partial k_y}\frac{\partial^2 \varepsilon_{\mathbf{k}}}{\partial k_x \partial k_y}\right\} \left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right) H_z}$$

C. Kittel, Quantum Theory of Solids.
J. M. Ziman, Principles of the theory of solids.
C. M. Hurd, The Hall effect in metals and alloys.

Quantum results based on Kubo formula: $\sigma_{xy} = \frac{4e^3 H}{3\pi c} \sum_k \int_{-\infty}^{\infty} dE \left(\frac{\partial f}{\partial E}\right) \left[\left(\frac{\partial \varepsilon}{\partial k_x}\right)^2 \frac{\partial^2 \varepsilon}{\partial k_y^2} - \frac{\partial \varepsilon}{\partial k_x}\frac{\partial \varepsilon}{\partial k_y}\frac{\partial^2 \varepsilon}{\partial k_x \partial k_y}\right] \cdot (\mathrm{Im}\, G^R(k, E))^3$

H. Fukuyama, Prog. Theor. Phys. **42**, 1284 (1969).

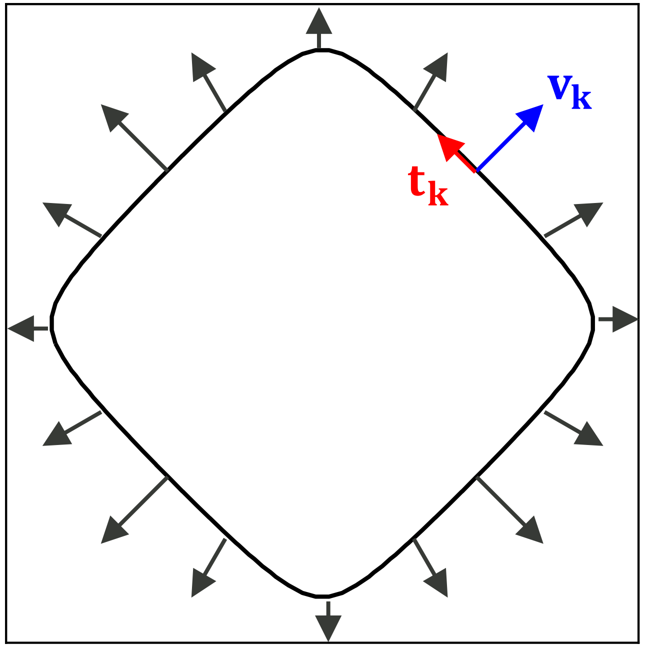
Geometrical aspect of Hall conductivity

N. P. Ong, Phys. Rev. B **43**, 193 (1991).

$$\frac{\sigma_{xy}}{H_z} = \frac{1}{(2\pi)^3}\int d^3k\, \frac{q_e^3\tau_{\mathbf{k}}^2}{c\hbar^4}\left\{\left(\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_x}\right)^{\mathbf{2}}\frac{\partial^2\varepsilon_{\mathbf{k}}}{\partial k_y^2} - \frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_x}\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_y}\frac{\partial^2\varepsilon_{\mathbf{k}}}{\partial k_x\partial k_y}\right\}\left(-\frac{\partial f_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}}\right)$$

$$\frac{1}{\hbar^3}\left\{\left(\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_x}\right)^{\mathbf{2}}\frac{\partial^2\varepsilon_{\mathbf{k}}}{\partial k_y^2} - \frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_x}\frac{\partial\varepsilon_{\mathbf{k}}}{\partial k_y}\frac{\partial^2\varepsilon_{\mathbf{k}}}{\partial k_x\partial k_y}\right\} = -v_{\mathbf{k}x}(\mathbf{v}_{\mathbf{k}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\nabla}_{\mathbf{k}}\, v_{\mathbf{k}y}$$

$$\frac{1}{(2\pi)^2}\int dk_x\, dk_y\left(-\frac{\partial f_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}}\right) = \frac{1}{(2\pi)^2}\int dk_t\,\frac{1}{\hbar|\mathbf{v}|}$$



2D

$$\frac{\sigma_{xy}}{H_z} = -\frac{1}{(2\pi)^2}\int dk_t\,\frac{q_e^3\tau_{\mathbf{k}}^2}{c\hbar}\frac{1}{\hbar|\mathbf{v}_{\mathbf{k}}|}v_{\mathbf{k}x}(\mathbf{v}_{\mathbf{k}}\times\hat{\mathbf{z}})\cdot\boldsymbol{\nabla}_{\mathbf{k}}\, v_{\mathbf{k}y} = -\frac{1}{(2\pi)^2}\int dk_t\,\frac{q_e^3\tau_{\mathbf{k}}^2}{c\hbar^2}v_{\mathbf{k}x}\,\mathbf{t}_{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\, v_{\mathbf{k}y}$$

$$= -\frac{1}{(2\pi)^2}\int dk_t\,\frac{q_e^3\tau_{\mathbf{k}}^2}{c\hbar^2}\left(v_{\mathbf{k}x}\mathbf{t}_{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\, v_{\mathbf{k}y} - v_{\mathbf{k}y}\mathbf{t}_{\mathbf{k}}\cdot\boldsymbol{\nabla}_{\mathbf{k}}\, v_{\mathbf{k}x}\right)\frac{1}{2} = \frac{1}{(2\pi)^2}\int(d\mathbf{v}\times\mathbf{v})\cdot\hat{\mathbf{z}}\,\frac{1}{2}\frac{q_e^3\tau_{\mathbf{k}}^2}{c\hbar^2} = \frac{1}{8\pi^2}\frac{q_e^3\tau^2}{c\hbar^2}\,S_{FS}$$

3D $$\frac{\sigma_{xy}}{H_z} = \frac{1}{16\pi^3}\frac{q_e^3\tau^2}{c\hbar^2}\int dk_z\, S_{z,FS}$$

$$\frac{\sigma_{xy}}{H_z} = \frac{1}{(2\pi)^2}\int(d\mathbf{l}\times\mathbf{l})\cdot\hat{\mathbf{z}}\,\frac{1}{2}\frac{q_e^3}{c\hbar^2} = \frac{1}{8\pi^2}\frac{q_e^3}{c\hbar^2}\,A$$

A: "Stokes" area

$\mathbf{l}_{\mathbf{k}} = \tau_{\mathbf{k}}\mathbf{v}_{\mathbf{k}}$

Hall conductivity and Hall coefficient R_H are intimately connected to the Fermi surface or volume.

Material specifics

J. R. Yates, X. Wang, D. Vanderbilt, and I. Souza, Phys. Rev. B **75**, 195121 (2007).

Include multi-band nature, but neglect interband contributions $\varepsilon_{\mathbf{k}} \rightarrow \varepsilon_{n\mathbf{k}} = \langle n\mathbf{k}|H_{\mathbf{k}}|n\mathbf{k}\rangle$

$$\frac{\sigma_{xy}}{H_z} = \frac{1}{(2\pi)^3}\int d^3k \sum_n \frac{q_e^3 \tau_{n\mathbf{k}}^2}{c}\left\{v_{n\mathbf{k}x}^{\mathbf{2}}\eta_{n\mathbf{k}yy} - v_{n\mathbf{k}x}v_{n\mathbf{k}y}\eta_{n\mathbf{k}xy}\right\}\left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right) \qquad \sigma_{\alpha\alpha} = \frac{1}{(2\pi)^3}\int d^3k \sum_n q_e^2\, \tau_{n\mathbf{k}}\, v_{n\mathbf{k}\alpha\alpha}^{\mathbf{2}}\left(-\frac{\partial f_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}}\right)$$

$$v_{n\mathbf{k}\alpha} = \frac{1}{\hbar}\left\langle n\mathbf{k}\middle|\frac{\partial H_{\mathbf{k}}}{\partial k_\alpha}\middle|n\mathbf{k}\right\rangle$$

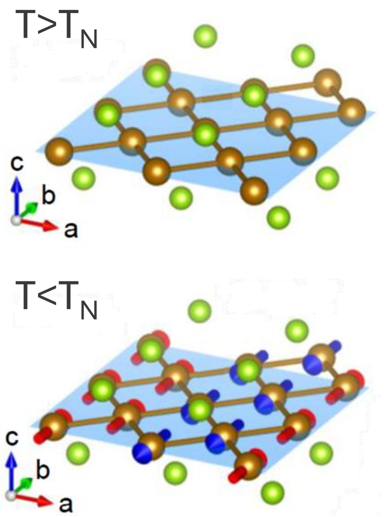
$$\eta_{n\mathbf{k}\alpha\beta} = \frac{1}{\hbar^2}\left\langle n\mathbf{k}\middle|\frac{\partial^2 H_{\mathbf{k}}}{\partial k_\alpha \partial k_\beta}\middle|n\mathbf{k}\right\rangle + \frac{1}{\hbar^2}\sum_m \frac{\left\langle n\mathbf{k}\middle|\frac{\partial H_{\mathbf{k}}}{\partial k_\alpha}\middle|m\mathbf{k}\right\rangle\left\langle m\mathbf{k}\middle|\frac{\partial H_{\mathbf{k}}}{\partial k_\beta}\middle|n\mathbf{k}\right\rangle + \left\langle n\mathbf{k}\middle|\frac{\partial H_{\mathbf{k}}}{\partial k_\beta}\middle|m\mathbf{k}\right\rangle\left\langle m\mathbf{k}\middle|\frac{\partial H_{\mathbf{k}}}{\partial k_\alpha}\middle|n\mathbf{k}\right\rangle}{\varepsilon_{n\mathbf{k}} - \varepsilon_{m\mathbf{k}}}$$

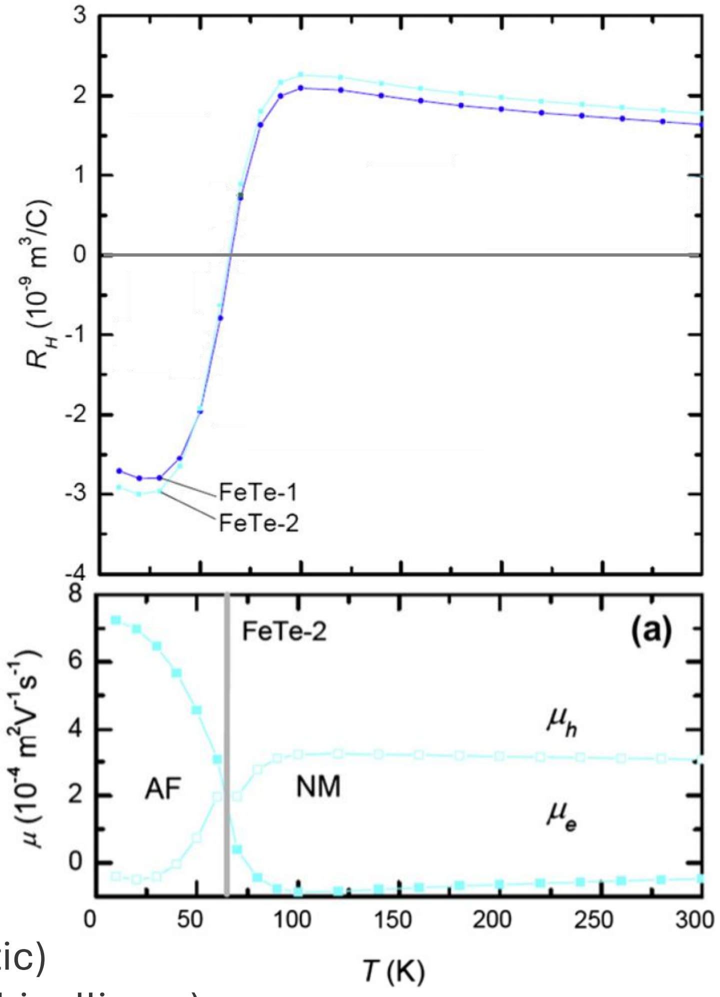
$H_{\mathbf{k}}$ and $|n\mathbf{k}\rangle$ can be obtained from DFT calculations via Wannier90

Need extra information for $\tau_{n\mathbf{k}}$

Material specifics: FeTe

I. Tsukada et al. Physica C **471**, 625 (2011).





Two-band model fitting

$$\sigma_{xx,yy} = |e|(\mu_h n_h + \mu_e n_e)$$

$$\sigma_{xy} = |e|(\mu_h^2 n_h - \mu_e^2 n_e)H_z$$

$$R_H = \frac{\mu_h^2 n_h - \mu_e^2 n_e}{|e|(\mu_h n_h + \mu_e n_e)^2}$$

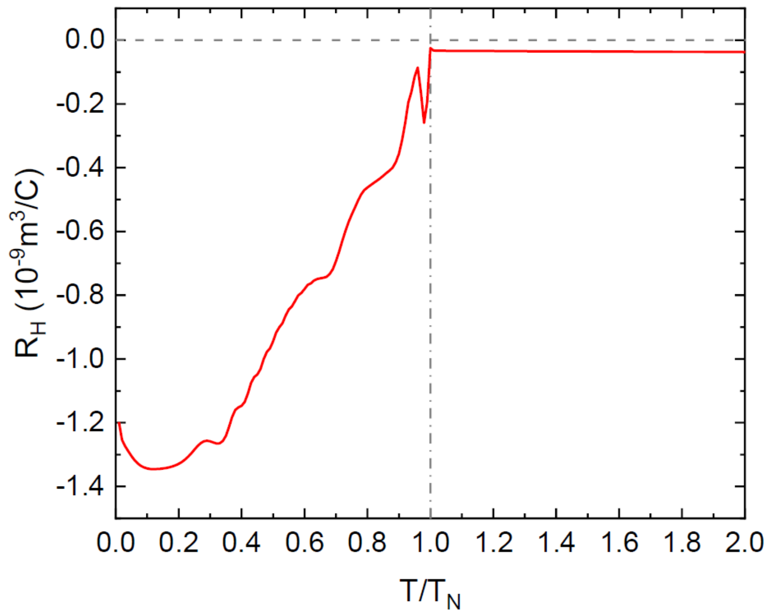
$n_e = n_h = 0.8$/cell (nonmagnetic)

$n_e = 0.43$/cell, $n_e = 0.41$/cell (bicollinear)

F. Ma et al. Phys. Rev. Lett. **102**, 177003 (2009).

High-temperature positive R_H from large μ_h or something else?

Theoretical result



S. Okamoto, A. Moreo, N. Nagaosa, and S. S. P. Parkin, arXiv:2604.11583.

Oak Ridge National Laboratory

July 13-15, 2026, Colorado State University

Hall effects and correlations in quantum materials

Satoshi Okamoto

Materials Science and Technology Division

U.S. Department of Energy

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FOR THE US DEPARTMENT OF ENERGY

Outline

Various Hall effects
- Ordinary Hall effect
- Integer quantum Hall effect
- Fractional quantum Hall effect
- Anomalous Hall effect
- Spin Hall effect

Ordinary Hall effect
- Semiclassical description
- Example of material specifics

Day 1

Anomalous Hall effect
- Berry curvature in momentum space
- Berry curvature in real space: topological Hall effect
- Two example

Day 2

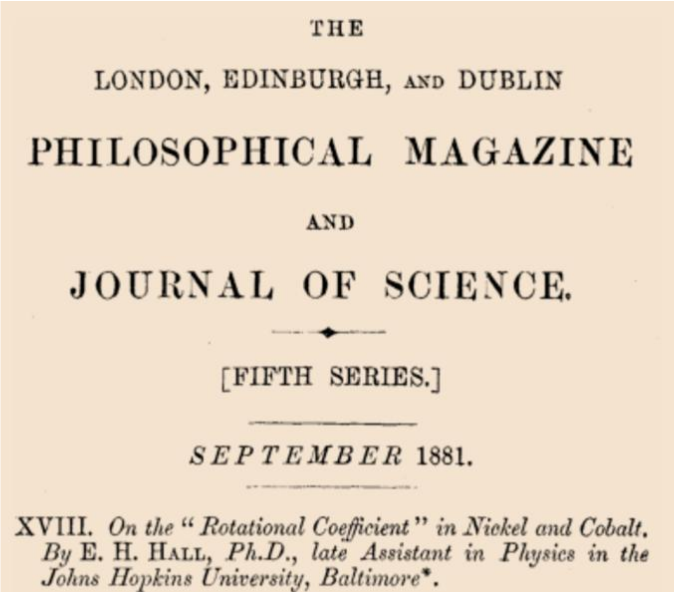
Spin Hall effect
- Intrinsic mechanisms
- Extrinsic mechanisms
- Phonon mechanisms
- Spin fluctuation mechanisms

Chern insulators
- Integer Chern insulators
- Fractional Chern insulators

Day 3

Anomalous Hall effect

Anomalous Hall effect



THE

LONDON, EDINBURGH, AND DUBLIN

PHILOSOPHICAL MAGAZINE

AND

JOURNAL OF SCIENCE.

[FIFTH SERIES.]

SEPTEMBER 1881.

XVIII. *On the "Rotational Coefficient" in Nickel and Cobalt.* *By* E. H. HALL, *Ph.D., late Assistant in Physics in the Johns Hopkins University, Baltimore**.

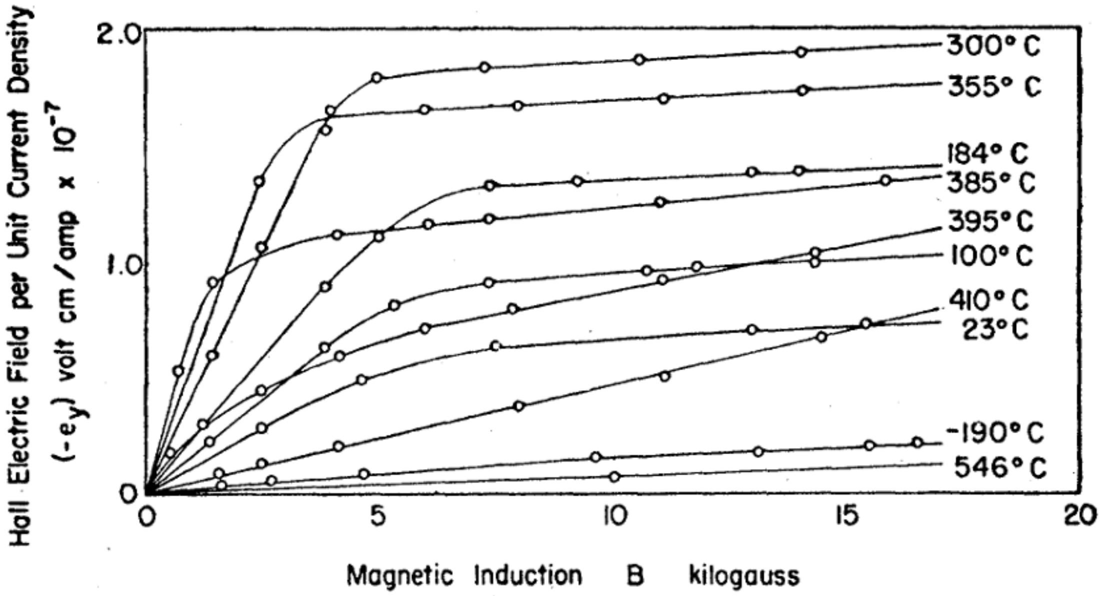


Fig. 1. Hall effect in nickel from the data of A. W. Smith.

A. W. Smith, Phys. Rev. **30**,1 (1910).
E. M. Pugh and N. Rostoker, Rev. Mod. Phys. **25**, 151 (1953).

$$E_y = R_0 H_z + R_1 M_z$$

Anomalous Hall effect was discovered by Hall himself only two years after the discovery of ordinary Hall effect. It took more than half a century to understand its mechanism (may not be fully yet).

Anomalous Hall effect: Berry curvature in momentum space

Intrinsic mechanism purely originating from band effects

Kubo formula

$\langle J_{\mathbf{q}\mu}(\omega)\rangle$ under perturbation of $H' = -\frac{1}{c} j_{-\mathbf{q}\nu} A_{\mathbf{q}\nu} e^{-i\omega t}$

$$\sigma_{\mu\nu}(\boldsymbol{q},\omega) = -\frac{1}{i\omega} K_{\mu\nu}(\boldsymbol{q},\omega)$$

$K_{\mu\nu}(\boldsymbol{q}, i\Omega)$: Response of electronic system against $\mathbf{A}_{\mathbf{q}} e^{-i\Omega\tau}$

Additional factor $1/i\omega$ after analytic continuation $i\Omega \to \hbar\omega$ is needed to convert $\mathbf{A}$ to $\mathbf{E}$.

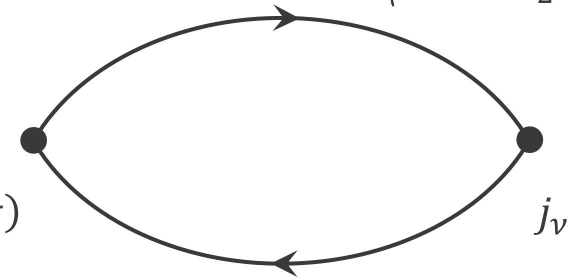
$i\Omega$: Bosonic Matsubara frequency

$$K_{\mu\nu}(\boldsymbol{q}, i\Omega) = -\int_0^{\beta=1/T} d\tau\, e^{i\Omega\tau} \langle T_\tau J_{\mathbf{q}\mu}(\tau) J_{-\mathbf{q}\nu}(0)\rangle$$

$$= -\boldsymbol{q}_e^2 \int_0^{\beta=1/T} d\tau\, e^{i\Omega\tau} \sum_{ll'l''l'''\mathbf{k}\boldsymbol{k}'} v_{\mu ll'\mathbf{k}}\, v_{\mu l''l'''\mathbf{k}'} \left\langle T_\tau \psi^\dagger_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(\tau)\, \psi_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau) \psi^\dagger_{l''\,\mathbf{k}'+\frac{1}{2}\mathbf{q}}(0)\, \psi_{l'''\,\mathbf{k}'-\frac{1}{2}\mathbf{q}}(0)\right\rangle$$

$$-\left\langle T_\tau \psi_{l'\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(0) \psi^\dagger_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(\tau)\right\rangle = \delta_{ll'} G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(-\tau)$$

$\boxed{K_{\mu\nu}(\boldsymbol{q}, i\Omega)}$



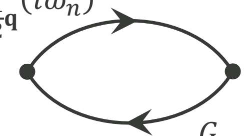
$$j_{\mu\mathbf{q}}(\tau) = q_e \sum_{ll'\mathbf{k}} v_{\mu ll'\mathbf{k}}\, \psi^\dagger_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(\tau)\, \psi_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau)$$

$$j_{\nu\,-\mathbf{q}}(0) = q_e \sum_{ll'\mathbf{k}} \mathbf{v}_{\nu\, ll'\mathbf{k}}\, \psi^\dagger_{l\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(0)\, \psi_{l'\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(0)$$

$$-\left\langle T_\tau \psi_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau) \psi^\dagger_{l\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(0)\right\rangle = \delta_{ll'} G_{l\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau)$$

Anomalous Hall effect: Berry curvature in momentum space

$$K_{\mu\nu}(\boldsymbol{q}, i\Omega) = q_e^2 \int_0^\beta d\tau\, e^{i\Omega\tau} \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}}\, G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau)\, G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(-\tau)$$



$$= q_e^2 \int_0^\beta d\tau\, e^{i\Omega\tau} \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}}\, T^2 \sum_{nn'} e^{-i\omega_n\tau}\, G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(i\omega_n) e^{i\omega_{n'}\tau} G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(i\omega_{n'})$$

$$\omega_n = (2n+1)\pi T$$

$$G_{\mathbf{k}}(\tau) = T \sum_n e^{-i\omega_n\tau}\, G_{\mathbf{k}}(i\omega_n)$$

$$= q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}}\, T \sum_n G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(i\omega_n + i\Omega)\; G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(i\omega_n)$$

$$= q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}} \frac{f\left(\varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}\right) - f\left(\varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}\right)}{i\Omega + \varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}} - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}}$$

$$G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(i\omega_n + i\Omega)\, G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(i\omega_n) = \frac{1}{i\omega_n + i\Omega - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}} \frac{1}{i\omega_n - \varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}}$$

$$= \frac{1}{i\Omega + \varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}} - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}} \left\{ \frac{1}{i\omega_n - \varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}} - \frac{1}{i\omega_n + i\Omega - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}} \right\}$$

$$K_{\mu\nu}(\boldsymbol{q}, \hbar\omega) = q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}} \frac{f\left(\varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}\right) - f\left(\varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}\right)}{\hbar\omega + \varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}} - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}} = -q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}} \frac{f\left(\varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}\right) - f\left(\varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}\right)}{\left(\varepsilon_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}} - \varepsilon_{l'\mathbf{k}+\frac{1}{2}\mathbf{q}}\right)^2} \hbar\omega$$

TKKN

$$\sigma_{\mu\nu} = -\frac{1}{i\omega} K_{\mu\nu}(\hbar\omega) = -i\hbar\, q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}} \frac{f(\varepsilon_{l\,\mathbf{k}}) - f(\varepsilon_{l'\mathbf{k}})}{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2} = -i\hbar\, q_e^2 \sum_{\mathbf{k},ll'} \frac{f(\varepsilon_{l\,\mathbf{k}})}{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2} \left(v_{\mu l l' \mathbf{k}} v_{\nu l' l\,\mathbf{k}} - v_{\nu l l' \mathbf{k}} v_{\mu l' l\,\mathbf{k}} \right)$$

All occupied states contribute to Hall conductivity.

Cf. States at the Fermi surface contribute to longitudinal electrical conductivity.

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Anomalous Hall effect: Berry curvature in momentum space

$$\sigma_{\mu\nu} = -i\hbar\, q_e^2 \sum_{\mathbf{k},ll'} \frac{f(\varepsilon_{l\,\mathbf{k}})}{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2} \left(v_{\mu l l'\mathbf{k}} v_{\nu l' l\,\mathbf{k}} - v_{\nu l l'\mathbf{k}} v_{\mu l' l\,\mathbf{k}} \right)$$

$$v_{\mu l l'\mathbf{k}} v_{\nu l' l\,\mathbf{k}} - v_{\nu l l'\mathbf{k}} v_{\mu l' l\,\mathbf{k}} = \frac{1}{\hbar^2}\left\{ \left\langle l\,\mathbf{k} \middle| \frac{\partial H_{\mathbf{k}}}{\partial k_\mu} \middle| l'\mathbf{k} \right\rangle \left\langle l'\mathbf{k} \middle| \frac{\partial H_{\mathbf{k}}}{\partial k_\nu} \middle| l\,\mathbf{k} \right\rangle - \left\langle l\,\mathbf{k} \middle| \frac{\partial H_{\mathbf{k}}}{\partial k_\nu} \middle| l'\mathbf{k} \right\rangle \left\langle l'\mathbf{k} \middle| \frac{\partial H_{\mathbf{k}}}{\partial k_\mu} \middle| l\,\mathbf{k} \right\rangle \right\}$$

$$\mathbf{v}_{\mathbf{k}} = \frac{1}{\hbar}\frac{\partial H_{\mathbf{k}}}{\partial \mathbf{k}}$$

$$= \frac{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2}{\hbar^2} \left\{ \langle \partial_\mu\, l\,\mathbf{k} | l'\mathbf{k} \rangle \langle l'\,\mathbf{k} | \partial_\nu l\,\mathbf{k} \rangle - \langle \partial_\nu\, l\,\mathbf{k} | l'\mathbf{k} \rangle \langle l'\,\mathbf{k} | \partial_\mu l\,\mathbf{k} \rangle \right\}$$

$$\sigma_{xy} = -i\frac{q_e^2}{\hbar} \sum_{\mathbf{k},ll'} f(\varepsilon_{l\,\mathbf{k}}) \left\{ \langle \partial_x\, l\,\mathbf{k} | l'\mathbf{k} \rangle \langle l'\,\mathbf{k} | \partial_y l\,\mathbf{k} \rangle - \langle \partial_y\, l\,\mathbf{k} | l'\mathbf{k} \rangle \langle l'\,\mathbf{k} | \partial_x l\,\mathbf{k} \rangle \right\}$$

$$\langle l\,\mathbf{k} | \partial_\alpha H(\mathbf{k}) | l'\,\mathbf{k} \rangle = \langle l\,\mathbf{k} | \varepsilon_{l'\,\mathbf{k}} | \partial_\alpha l'\,\mathbf{k} \rangle - \langle l\,\mathbf{k} | \varepsilon_{l\,\mathbf{k}} | \partial_\alpha l'\mathbf{k} \rangle$$
$$= -(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}}) \langle l\,\mathbf{k} | \partial_\alpha l'\mathbf{k} \rangle = (\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\,\mathbf{k}}) \langle \partial_\alpha\, l\,\mathbf{k} | l'\mathbf{k} \rangle$$

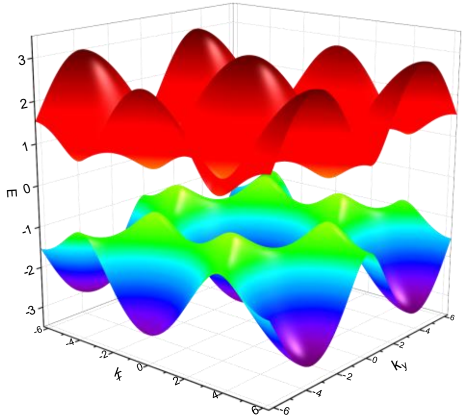
$$= -i\frac{q_e^2}{\hbar} \sum_{\mathbf{k},l} f(\varepsilon_{l\,\mathbf{k}}) \left\{ \langle \partial_x\, l\,\mathbf{k} | \partial_y l\,\mathbf{k} \rangle - \langle \partial_y\, l\,\mathbf{k} | \partial_x l\,\mathbf{k} \rangle \right\}$$

$$\boldsymbol{A}_{ll\mathbf{k}} = i\langle l\,\mathbf{k} | i\boldsymbol{\nabla}_{\mathbf{k}} | l\,\mathbf{k} \rangle$$

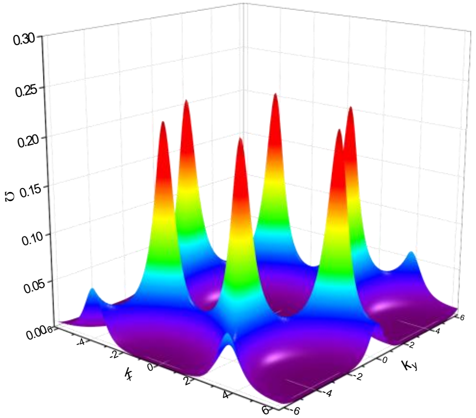
$$= -\frac{q_e^2}{\hbar} \sum_{\mathbf{k},l} f(\varepsilon_{l\,\mathbf{k}}) \left(\boldsymbol{\nabla}_{\boldsymbol{k}} \times \boldsymbol{A}_{ll\mathbf{k}} \right)_z = -\frac{q_e^2}{\hbar} \sum_{\mathbf{k},l} f(\varepsilon_{l\,\mathbf{k}})\, \Omega_{\boldsymbol{l}\,\boldsymbol{z}}$$

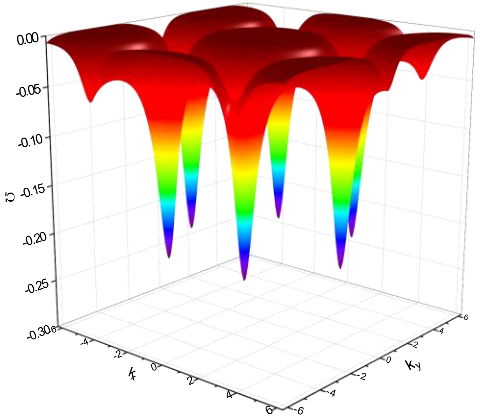
σ_{xy} is given by the momentum integral of Berry curvature $\Omega_{\boldsymbol{l}\,\boldsymbol{z}}$

Example: Haldane model
Phys. Rev. Lett. **61**, 2015 (1988).



Dispersion



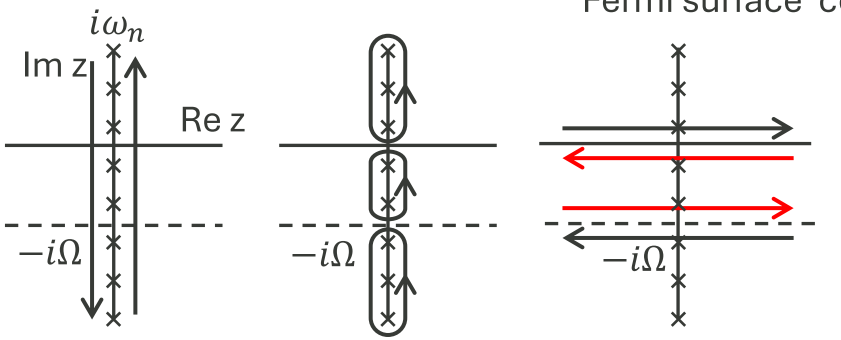


Berry curvature of top band　　　bottom band

Anomalous Hall effect: Fermi surface contribution

$$K_{\mu\nu}(\boldsymbol{q}, i\Omega) = q_e^2 \int_0^\beta d\tau\, e^{i\Omega\tau} \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\, \mathbf{k}}\, G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(\tau)\, G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(-\tau)$$

$$= q_e^2 \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\, \mathbf{k}}\, T \sum_n G_{l'\,\mathbf{k}+\frac{1}{2}\mathbf{q}}(i\omega_n + i\Omega)\, G_{l\,\mathbf{k}-\frac{1}{2}\mathbf{q}}(i\omega_n)$$



Fermi surface contribution $= -\int \frac{dx}{2\pi i} f(x)\left[-G^R_{l'\,\mathbf{k}}(x+\hbar\omega) G^A_{l\,\mathbf{k}}(x) + G^R_{l'\,\mathbf{k}}(x) G^A_{l\,\mathbf{k}}(x-\hbar\omega)\right]$

$$= -\int \frac{dx}{2\pi i}\left[-f(x) G^R_{l'\,\mathbf{k}}(x+\hbar\omega) G^A_{l\,\mathbf{k}}(x) + f(x+\hbar\omega) G^R_{l'\,\mathbf{k}}(x+\hbar\omega) G^A_{l\,\mathbf{k}}(x)\right]$$

$$= -\int \frac{dx}{2\pi i}\left(-f(x) + f(x+\hbar\omega)\right) G^R_{l'\,\mathbf{k}}(x+\hbar\omega) G^A_{l\,\mathbf{k}}(x)$$

$$= -\int \frac{dx}{2\pi i}\left(\frac{\partial f(x)}{\partial x}\right) \hbar\omega\, G^R_{l'\,\mathbf{k}}(x) G^A_{l\,\mathbf{k}}(x)$$

Fermi surface contribution to $\sigma_{\mu\nu} = -\frac{1}{i\omega} K_{\mu\nu}(\hbar\omega) = q_e^2 \hbar \int \frac{dx}{2\pi}\left(-\frac{\partial f(x)}{\partial x}\right) \sum_{\mathbf{k},ll'} v_{\mu l l' \mathbf{k}} v_{\nu l' l\, \mathbf{k}}\, G^R_{l'\,\mathbf{k}}(x) G^A_{l\,\mathbf{k}}(x)$

$$\sigma_{xy} = -\frac{nec}{H} + \Delta\sigma_{xy}$$

$$\Delta\sigma^{(2)}_{\mathrm{XY}} = \frac{e^2}{4\pi^2\hbar}\int\left(-\frac{\partial f}{\partial E}\right) \mathrm{d}E \sum_{N_1} \sum_{\pm} (\pm i)\, \xi^x_{N_1 N_1\pm 1}(E+i0, E-i0) \times G_{N_1}(E+i0) G_{N_1\pm 1}(E-i0)\, \Xi^x_{N_1\pm 1 N_1}(E-i0, E+i0)\,.$$

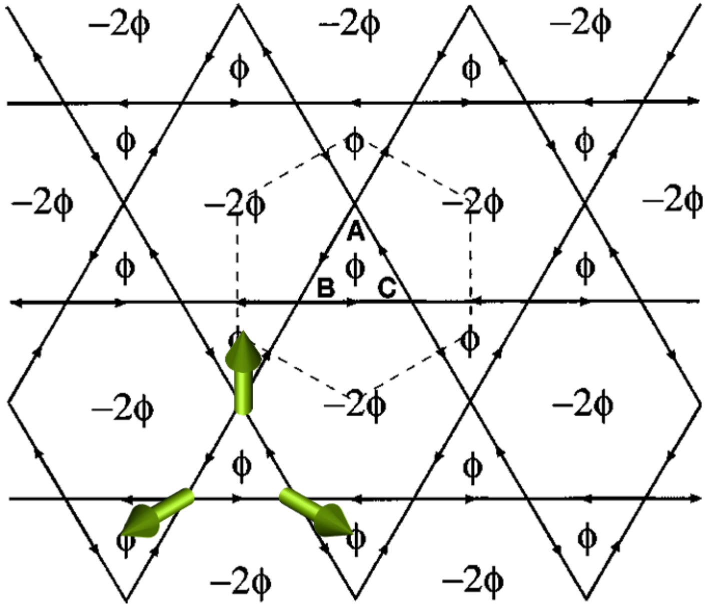
T. Ando, Y. Matsumoto, and Y. Uemura, J. Phys. Soc. Jpn. **39**, 279 (1975).

Berry curvature in real space: topological Hall effect

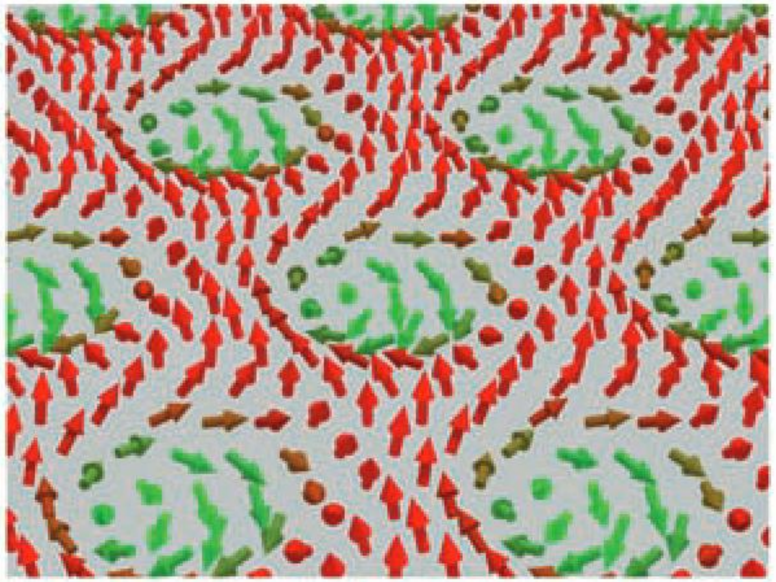
$$\chi_{lij} = \mathbf{S}_l \cdot (\mathbf{S}_i \times \mathbf{S}_j)$$

$$\sigma^{THE} \propto \langle \chi_{lij} \rangle$$

K. Ohgushi, S. Murakami, and N. Nagaosa, Phys. Rev. B **62**, R6065(R) (2000).
P. Bruno, V. K. Dugaev, and M. Taillefumier, Phys. Rev. Lett. **93**, 096806 (2004).
S. Mühlbauer, B. Binz, F. Jonietz, C. Pfleiderer, A. Rosch, A. Neubauer, R. Georgii, and P. Böni, Science **323**, 915 (2009).
M. Lee, W. Kang, Y. Onose, Y. Tokura, and N. Ong, Phys. Rev. Lett. **102**, 186601 (2009).
A. Neubauer, C. Pfleiderer, B. Binz, A. Rosch, R. Ritz, P. G. Niklowitz, and P. Böni, Phys. Rev. Lett. **102**, 186602 (2009).
S. D. Yi, S. Onoda, N. Nagaosa, and J. H. Han, Phys. Rev. B **80**, 054416 (2009).

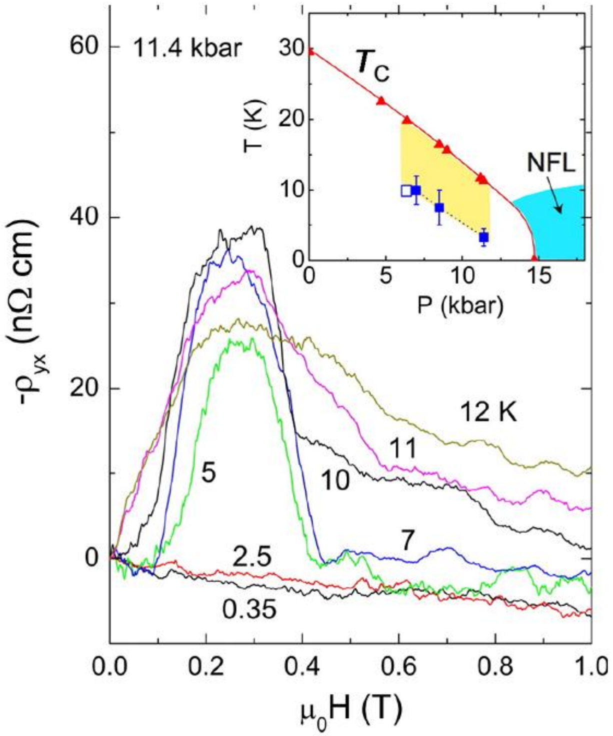


Ohgushi et al. Phys. Rev. B **62**, R6065(R) (2000).



Mühlbauer et al. Science **323**, 915 (2009).

MnSi



Lee et al. Phys. Rev. Lett. **102**, 186601 (2009).

Spin chirality $\chi_{lij} = \mathbf{S}_l \cdot (\mathbf{S}_i \times \mathbf{S}_j)$ could be another source of “anomalous” Hall effect.
The anomalous Hall effect induced by nonzero spin chirality is called topological Hall effect.

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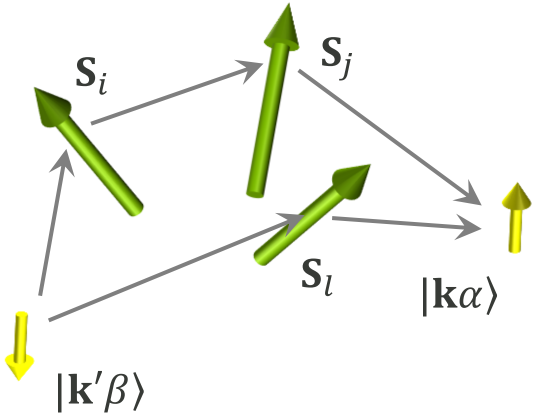
Berry curvature in real space: topological Hall effect

H. Ishizuka and N. Nagaosa, Science Advances **4**, eaap9962 (2018).

$$H_0 = \sum_{\mathbf{k}\alpha} \frac{\hbar^2 k^2}{2m} \varepsilon_{\mathbf{k}} c^\dagger_{\mathbf{k}\alpha} c_{\mathbf{k}\alpha}$$

$$H_K = J \sum_i \mathbf{S}_i \cdot c^\dagger_{i\alpha} \boldsymbol{\sigma}_{\alpha\beta} c_{i\beta}$$



Boltzmann equation again

$$-\frac{q_e}{\hbar}\mathbf{E}\cdot\hbar\mathbf{v}_{\mathbf{k}}\frac{\partial f^0_{\mathbf{k}}}{\partial \varepsilon_{\mathbf{k}}} + \left.\frac{\partial f_{\mathbf{k}\alpha}}{\partial t}\right|_{scatt} = 0 \qquad \left.\frac{\partial f_{\mathbf{k}\alpha}}{\partial t}\right|_{scatt} = -\frac{1}{\tau_{\mathbf{k}}} g_{\mathbf{k}\alpha} + \sum_{\mathbf{k}'\beta} W_{\mathbf{k}'\beta\to\mathbf{k}\alpha} g_{\mathbf{k}'\beta}$$

$$W_{\mathbf{k}'\beta\to\mathbf{k}\alpha} = \left|F^{(1)}_{\alpha\beta}(\mathbf{k},\mathbf{k}') + F^{(2)}_{\alpha\beta}(\mathbf{k},\mathbf{k}')\right|^2 \delta(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})$$

$$F^{(1)}_{\alpha\beta}(\mathbf{k},\mathbf{k}') \sim -J\sum_l \mathbf{S}_l\cdot\boldsymbol{\sigma}_{\alpha\beta}\, e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_l}$$

$$F^{(2)}_{\alpha\beta}(\mathbf{k},\mathbf{k}') \sim -iJ^2 m \sum_{i\neq j} \boldsymbol{\sigma}_{\alpha\beta}\cdot\mathbf{S}_j\times\mathbf{S}_i \frac{e^{-i\mathbf{k}\cdot\mathbf{R}_j + i\mathbf{k}'\cdot\mathbf{R}_i}}{|\mathbf{R}_j - \mathbf{R}_i|}$$

$$W_{\mathbf{k}'\beta\to\mathbf{k}\alpha} \sim -J^3 m \sum_{l,i\neq j} (\mathbf{S}_l\cdot\boldsymbol{\sigma}_{\alpha\beta})\{\boldsymbol{\sigma}_{\beta\alpha}\cdot(\mathbf{S}_i\times\mathbf{S}_j)\}\frac{e^{-i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{R}_l} e^{-i\mathbf{k}'\cdot\mathbf{R}_j + i\mathbf{k}\cdot\mathbf{R}_i}}{|\mathbf{R}_j - \mathbf{R}_i|}\delta(\varepsilon_{\mathbf{k}} - \varepsilon_{\mathbf{k}'})$$

$$g_{\mathbf{k}\alpha} = \tau_{\mathbf{k}}\frac{q_e}{\hbar}\mathbf{E}\cdot\hbar\mathbf{v}_{\mathbf{k}}\frac{\partial f^0_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}} + \tau_{\mathbf{k}}\sum_{\mathbf{k}'\beta} W_{\mathbf{k}'\beta\to\mathbf{k}\alpha} g_{\mathbf{k}'\beta} \qquad g_{\mathbf{k}\alpha} \sim q_e\tau^2_{\mathbf{k}}\sum_{\mathbf{k}'\beta}\frac{\partial f^0_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}} W_{\mathbf{k}'\beta\to\mathbf{k}\alpha}\mathbf{v}_{\mathbf{k}}\cdot\mathbf{E}$$

$$\mathbf{E} = (0, E_y, 0) \qquad \sigma^{THE} = \frac{J_x}{E_y} = \frac{1}{E_y}\sum_{\alpha\,\mathbf{k}} q_e v_{x\mathbf{k}}\, g_{\mathbf{k}\alpha} = \sum_{\alpha\beta\,\mathbf{k}\mathbf{k}'} q_e^2 v_{x\mathbf{k}} v_{y\mathbf{k}}\frac{\partial f^0_{\mathbf{k}}}{\partial\varepsilon_{\mathbf{k}}} W_{\mathbf{k}'\beta\to\mathbf{k}\alpha}$$

When three spins form triangle

$$W_{\mathbf{k}'\beta\to\mathbf{k}\alpha} \propto \langle \mathbf{S}_l\cdot(\mathbf{S}_i\times\mathbf{S}_j)\rangle$$

$$\sigma^{THE} \propto \langle \chi_{lij}\rangle$$

Spin chirality $\chi_{lij} = \mathbf{S}_l\cdot(\mathbf{S}_i\times\mathbf{S}_j)$ indeed induce topological Hall effect σ^{THE}.

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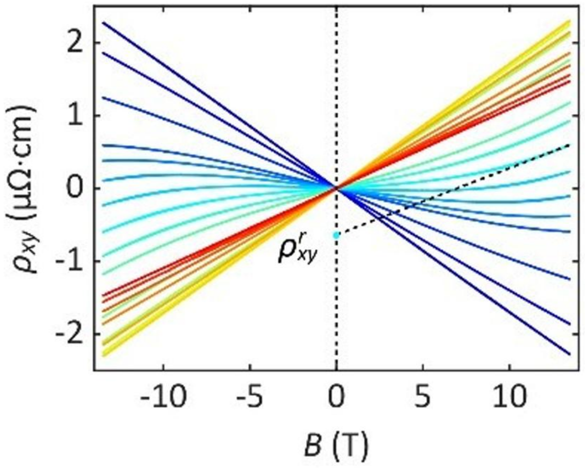
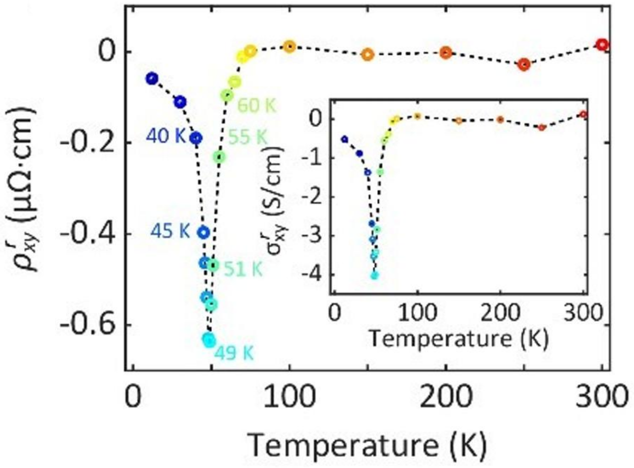
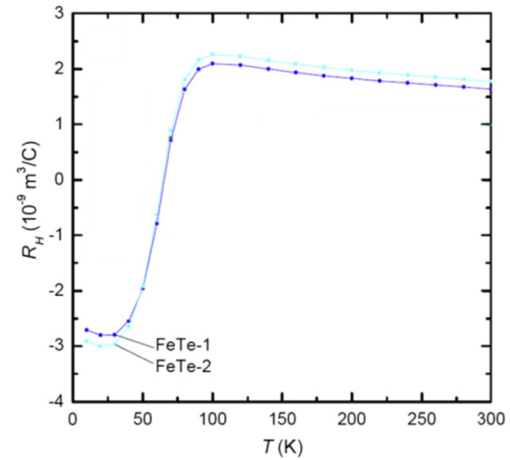
Field-induced intrinsic anomalous Hall effect in FeTe

R. Wang, C. Fang, I. Kostanovski, K. Xiao, F. Küster, J. Davern, N. Nagaosa, S. S. P. Parkin, arXiv:2604.12636.





Strong T dependence of AH conductivity

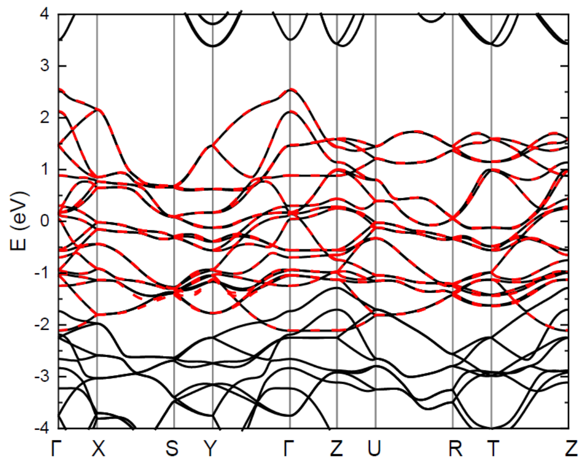


I. Tsukada et al. Physica C **471**, 625 (2011).

In addition to the previously reported anomaly in ordinary Hall effect, anomalous Hall effect shows unusual behavior.

Realistic model analysis is done using an effective model constructed based on DFT calculation.

S. Okamoto, A. Moreo, N. Nagaosa, S. S. P. Parkin, arXiv:2604.11583.



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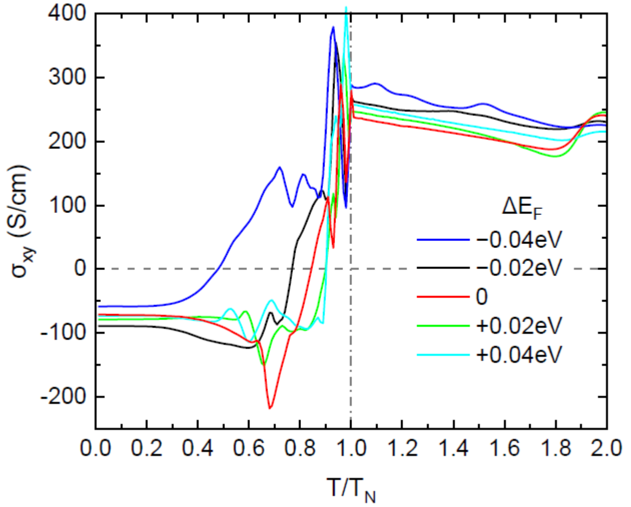
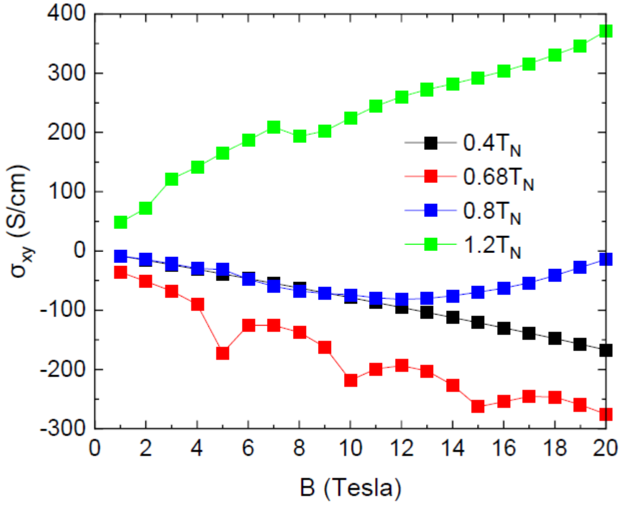
Field-induced intrinsic anomalous Hall effect in FeTe

Anomalous Hall conductivity at B=10 Tesla

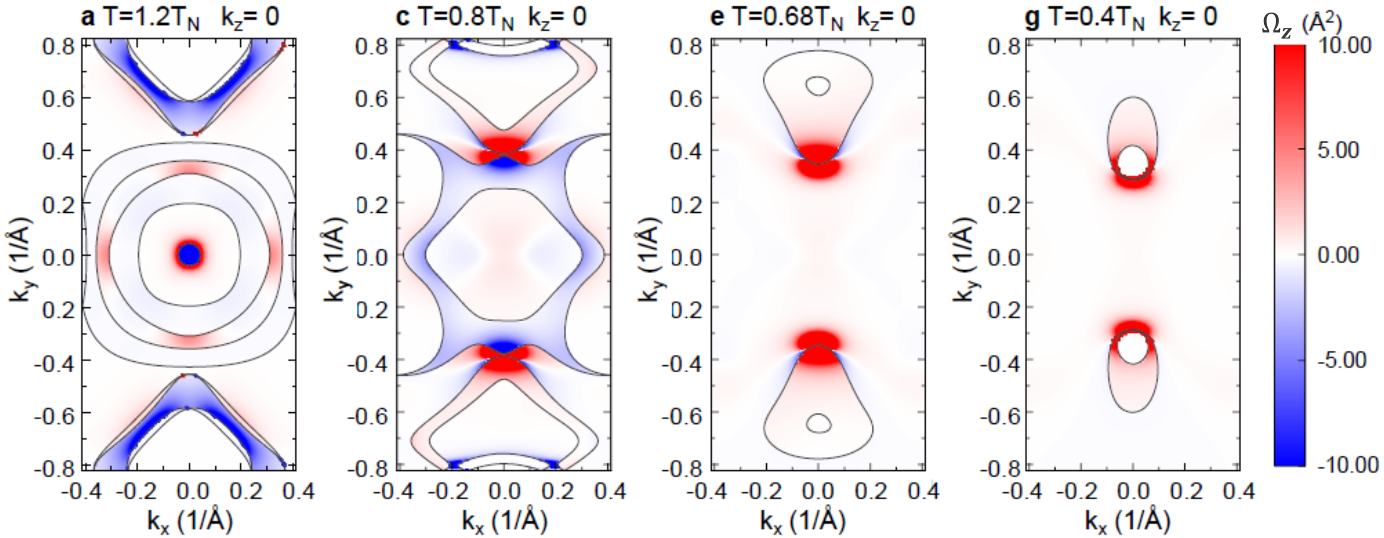


Anomalous Hall conductivity vs. B



σ_{xy} at B=10 Tesla has negative peak below T$_N$.
σ_{xy} increases with B at T>T$_N$ and decreases with B at T<T$_N$.
This may contribute to ordinary Hall response at T>T$_N$.

Temperature dependent Berry curvature at B=10 Tesla



Ω_z is largely negative at T>T$_N$, and largely positive at T<T$_N$, because antiferromagnetic ordering reorganizes band dispersions.

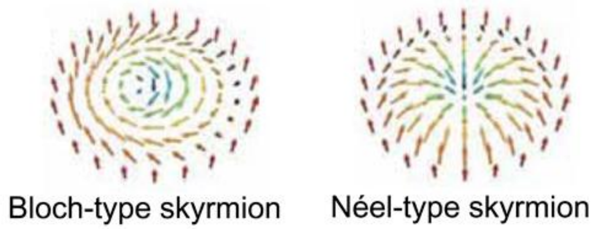
$$\sigma_{xy} = -\frac{q_e^2}{\hbar}\sum_{\mathbf{k},l} f(\varepsilon_{l\,\mathbf{k}})\,\Omega_{l\,z}$$

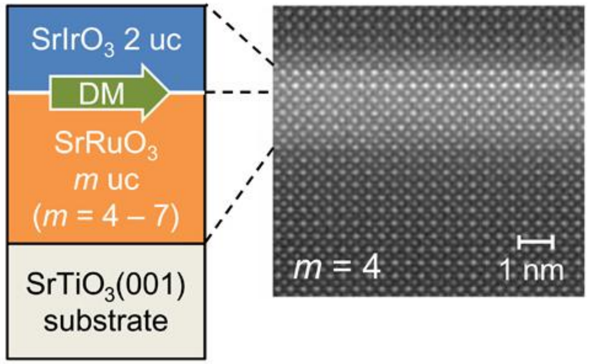
S. Okamoto, A. Moreo, N. Nagaosa, S. S. P. Parkin, arXiv:2604.11583.

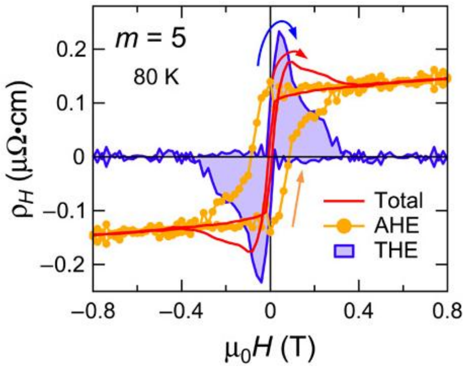
Topological Hall effect in heterostructures involving $SrIrO_3$ induced by Dzyaloshinskii-Moriya interaction

$SrRuO_3/SrIrO_3$ bilayers

Matsuno et al., Sci. Adv. **2**, e1600304(2016).

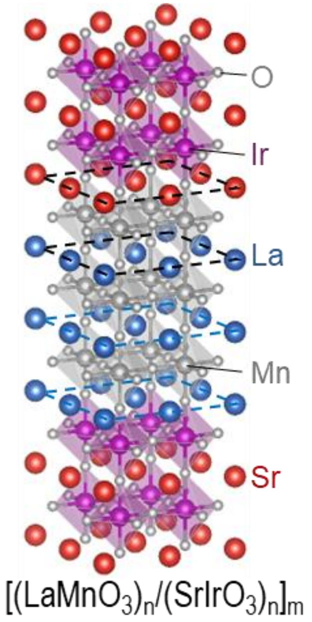


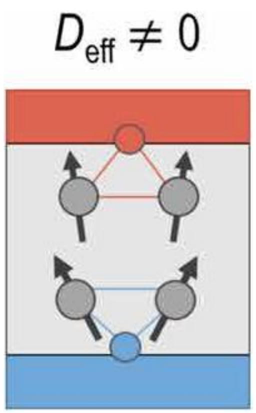




$LaMnO_3/SrIrO_3$ heterostructures

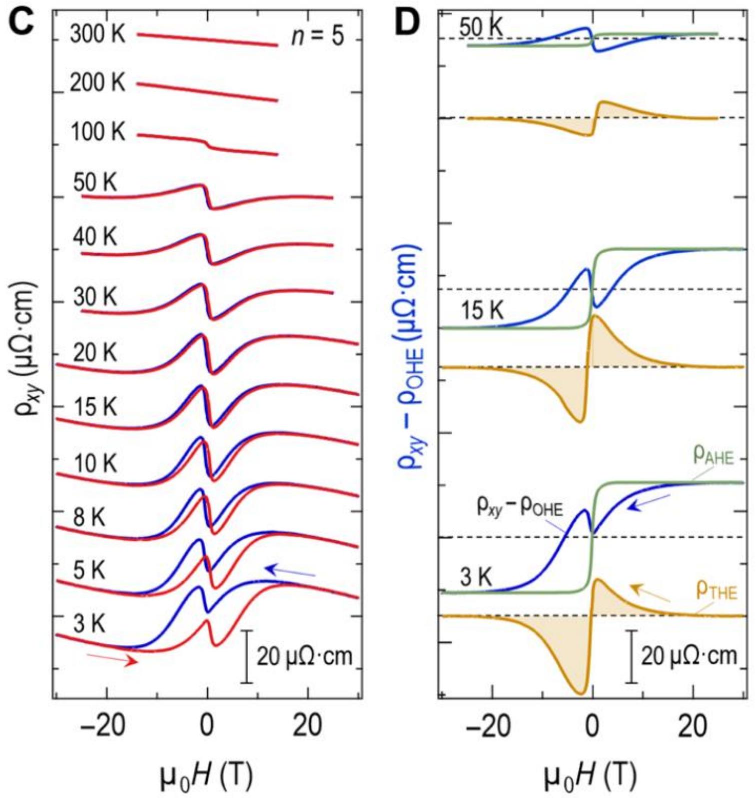
Skoropata, S.O., H.N. Lee et al., Sci. Adv. **6**, eaaz3902 (2020).





Controllable DMI

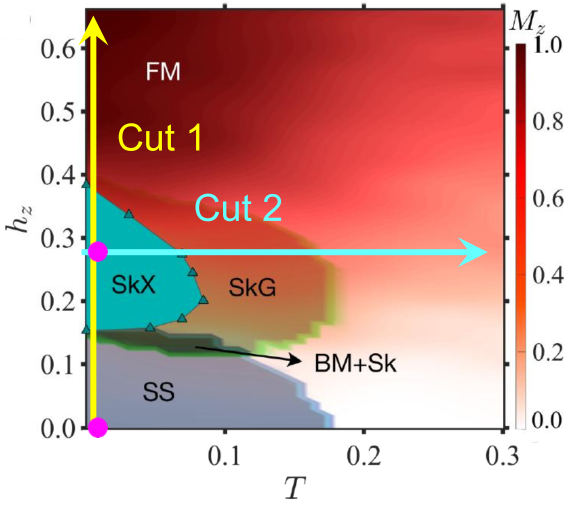
Antisymmetric spin interaction $\mathbf{D}_{ij} \cdot (\mathbf{S}_i \times \mathbf{S}_j)$



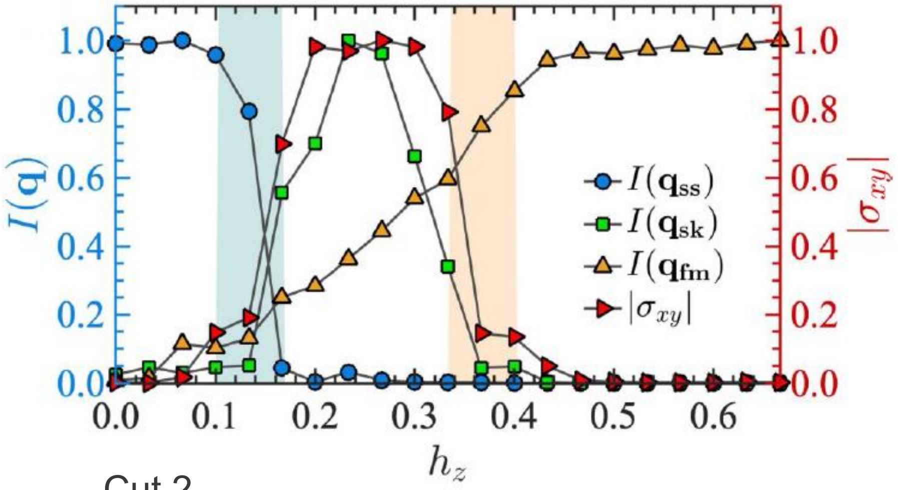
Topological Hall effect?

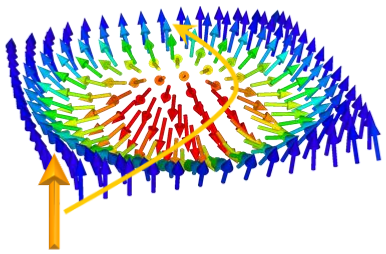
Chiral spin states are stabilized by Dzyaloshinskii-Moriya interaction.
Oxide heterostructures are rich playground to explore such phenomena.

Topological Hall effect in $(LaMnO_3)_m/(SrIrO_3)_n$ heterostructures: model analysis using double-exchange + DMI

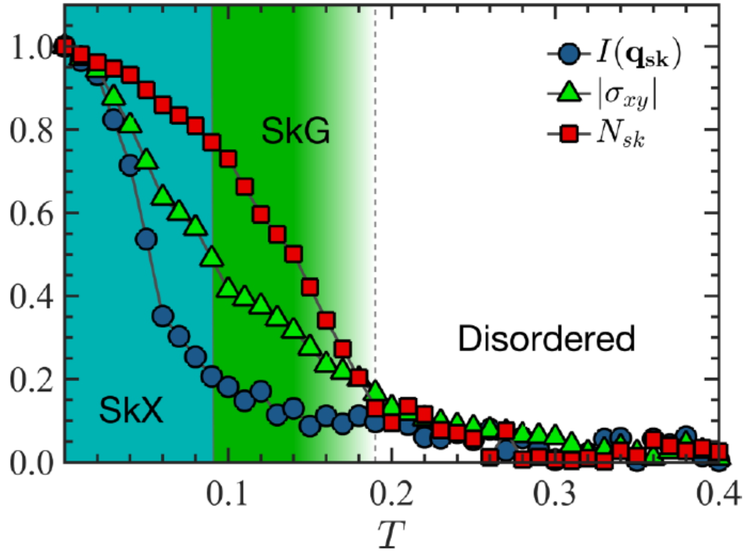


Cut 1





Cut 2



Hall conductivity σ_{xy} is controlled by the skyrmion density N_{sk} as a function of h and T.
σ_{xy} increases with decreasing T.
Supporting experimental observations.

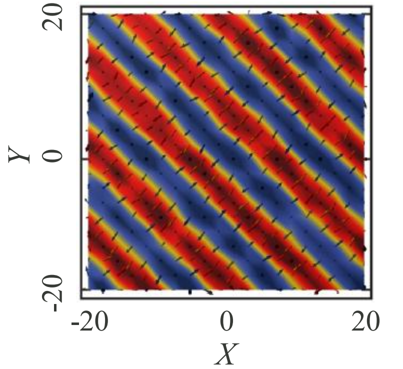
Hall conductivity

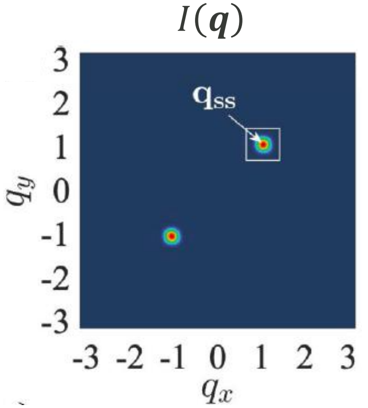
$$\sigma_{xy}(\omega) = \frac{1}{\hbar\omega}\int_0^\infty dt\, e^{i(\omega+i\delta)t}\langle[\hat{j}_x(t), \hat{j}_y(0)]\rangle$$

Skyrmion density

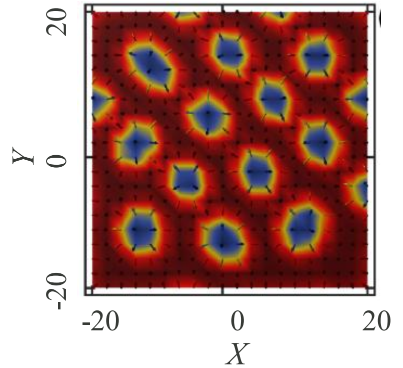
$$N_{sk} = \frac{1}{4\pi}\int dxdy\, \boldsymbol{S}\cdot\left(\frac{\partial \boldsymbol{S}}{\partial x}\times\frac{\partial \boldsymbol{S}}{\partial y}\right)$$

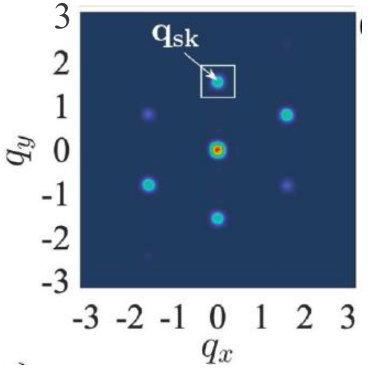
$h_z=0$: Spin spiral





$h_z=0.27$: Skyrmion crystal





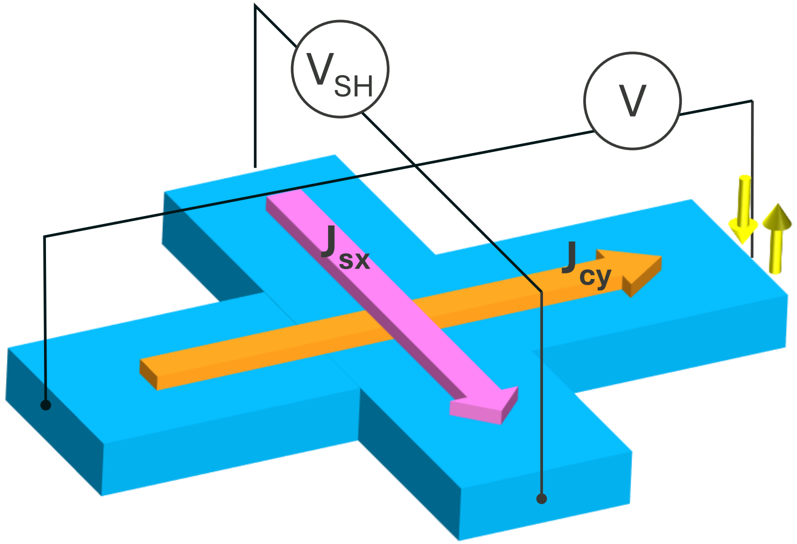
N. Mohanta, E. Dagotto, and S.O., Phys. Rev. B **100**, 064429 (2019).

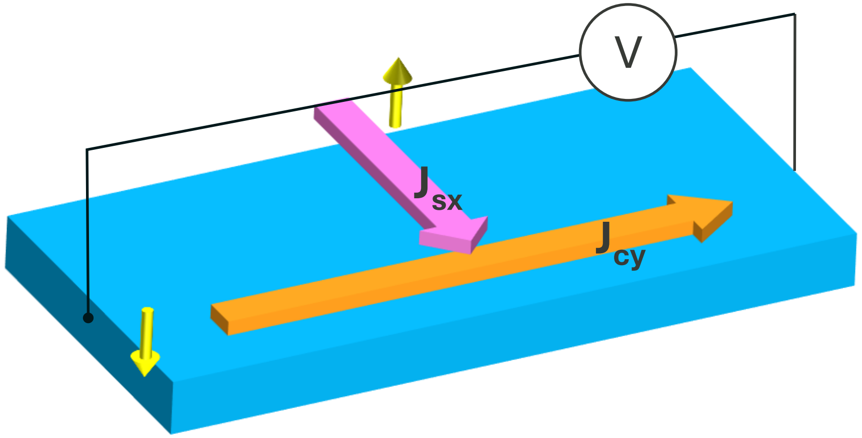
Spin Hall effect

Spin Hall effect

M.I. Dyakonov and V.I. Perel, JETP Lett. **13**, 467 (1971); Phys. Lett. **35A**, 459 (1971).
J. E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).

Inverse spin-Hall effect
E. Saitoh, M. Ueda, H. Miyajima, and G. Tatara, Appl. Phys. Lett. **88**, 182509 (2006).





Spin Hall effect could be useful and important for next generation electronics, spintronics and quantum applications, because it allows the manipulation of "spin" without using a magnetic field.

L. Zhu, D. C. Ralph, and R. A. Buhrman, Phys. Rev. Applied **10**, 031001 (2018). $\sigma^{sH} \sim 7.5 \times 10^5\ \Omega^{-1}\mathrm{m}^{-1}$ in Au-Pt alloys
C.-Y. Hu et al. ACS Appl. Electron. Mater. **4**,1099 (2022). $\sigma^{sH} \sim 6.5 \times 10^5\ (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ in Pt-Cr alloys
S. Rho et al. ACS Nano **19**, 38749 (2025). $\sigma^{sH} \sim 7.5 \times 10^5\ \Omega^{-1}\mathrm{m}^{-1}$ in topological semimetal $Bi_{1-x}Sb_x$
G. Ji et al. Nature Materials (2026). $\sigma^{sH} \sim 1.6 \times 10^7 (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ $PtCoO_2$

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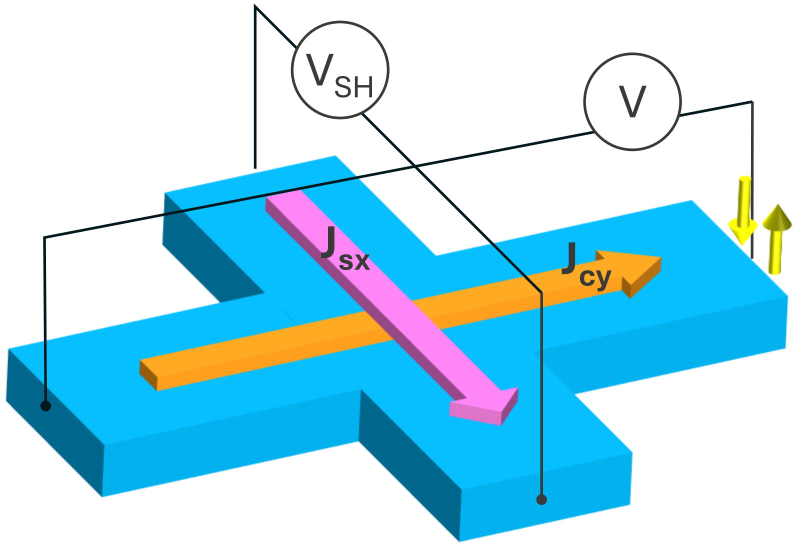
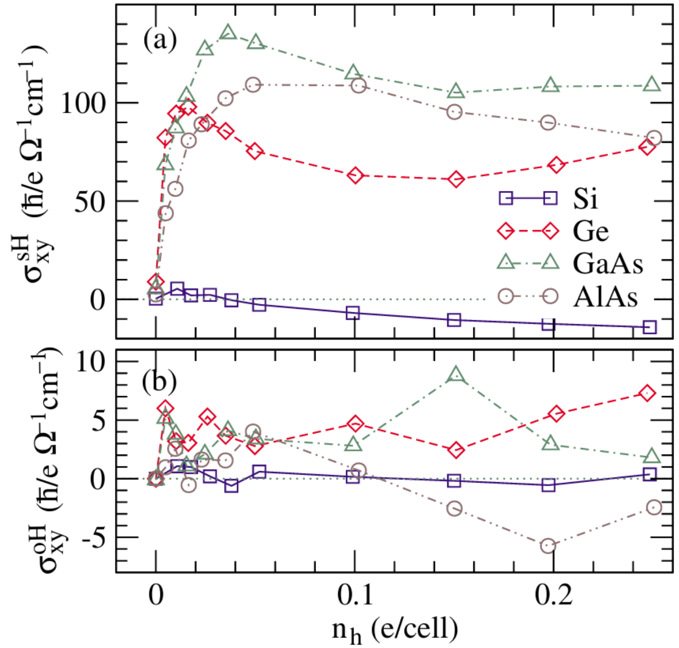
Spin Hall effect: Intrinsic mechanisms

J. Sinova, D. Culcer, Q. Niu, N. A. Sinitsyn, T. Jungwirth, and A. H. MacDonald, Phys. Rev. Lett. **92**, 126603 (2004).
G. Y. Guo, Y. Yao, and Q. Niu, Phys. Rev. Lett. **94**, 226601 (2005).
G. Y. Guo, S. Murakami, T.-W. Chen, and N. Nagaosa, Phys. Rev. Lett. **100**, 096401 (2008).
L. Wang, R. J. H. Wesselink, Y. Liu, Z. Yuan, K. Xia, and P. J. Kelly, Phys. Rev. Lett. **116**, 196602 (2016).

$$\sigma_{xy}^{sH} = -i\hbar q_e \sum_{\mathbf{k},ll'} \frac{f(\varepsilon_{l\,\mathbf{k}})}{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2} \left(v_{x\,ll'\mathbf{k}}^{spin,z} v_{yl'l\,\mathbf{k}} - v_{yll'\mathbf{k}} v_{x\,l'l\,\mathbf{k}}^{spin,z} \right)$$

$$\mathbf{v}_{\mathbf{k}}^{spin,z} = \frac{1}{2}\{s_z, \mathbf{v}_{\mathbf{k}}\}$$





Guo et al. Phys. Rev. Lett. **94**, 226601 (2005).

$$\sigma_{xy}^{oH} = -i\hbar q_e \sum_{\mathbf{k},ll'} \frac{f(\varepsilon_{l\,\mathbf{k}})}{(\varepsilon_{l\,\mathbf{k}} - \varepsilon_{l'\mathbf{k}})^2} \left(v_{x\,ll'\mathbf{k}}^{orbit,z} v_{yl'l\,\mathbf{k}} - v_{yll'\mathbf{k}} v_{x\,l'l\,\mathbf{k}}^{orbit,z} \right)$$

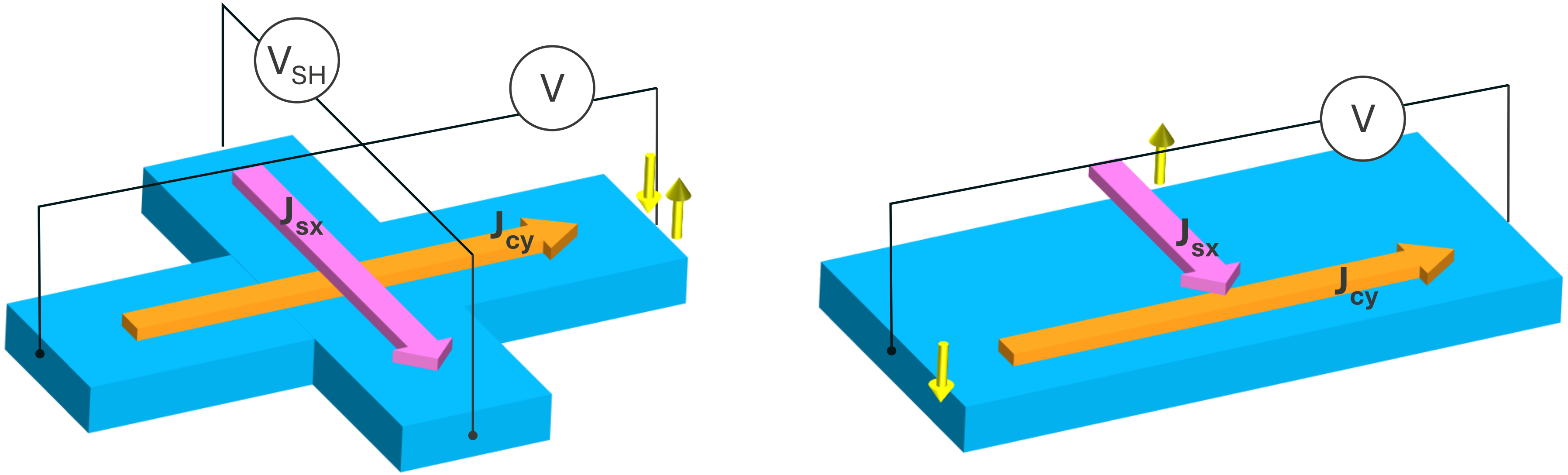
$$\mathbf{v}_{\mathbf{k}}^{orbit,z} = \frac{1}{2}\{L_z, \mathbf{v}_{\mathbf{k}}\}$$

Intrinsic spin (orbital) Hall conductivity could be computed in a similar way as intrinsic anomalous Hall conductivity, useful for material discovery.

Spin Hall effect

M.I. Dyakonov and V.I. Perel, JETP Lett. **13**, 467 (1971); Phys. Lett. **35A**, 459 (1971).
J. E. Hirsch, Phys. Rev. Lett. **83**, 1834 (1999).

Inverse spin-Hall effect
E. Saitoh, M. Ueda, H. Miyajima, and G. Tatara, Appl. Phys. Lett. **88**, 182509 (2006).



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C.-Y. Hu et al. ACS Appl. Electron. Mater. **4**,1099 (2022). $\sigma^{sH} \sim 6.5 \times 10^5\ (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ in Pt-Cr alloys
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G. Ji et al. Nature Materials (2026). $\sigma^{sH} \sim 1.6 \times 10^7 (\hbar/e)\ \Omega^{-1}\mathrm{m}^{-1}$ $PtCoO_2$

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Extrinsic mechanisms for spin Hall effect and anomalous Hall effect: impurities

W.-K. Tse and S. Das Sarma, Phys. Rev. Lett. **96**, 056601 (2006).

Spin-orbit coupling is essential: spin-dependent scattering

$$H_{SOC} = \frac{1}{2m^2c^2}\frac{1}{r}\frac{dV(\mathbf{r})}{dr}\mathbf{L}\cdot\mathbf{S} = \frac{\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot\{(\nabla_\mathbf{r}V(\mathbf{r}))\times\mathbf{p}\}$$

$$\mathbf{L} = \mathbf{r}\times\mathbf{p} \qquad \mathbf{p} = -i\hbar\nabla_\mathbf{r}$$

$$\mathbf{S} = \frac{1}{2}\hbar\boldsymbol{\sigma}$$

For example, L. I. Schiff, Quantum Mechanics

2nd quantization

$$H_{SOC} = \frac{\hbar}{4m^2c^2}\int d\mathbf{r}\,\psi^*_{\mathbf{k}'}(\mathbf{r})\boldsymbol{\sigma}\cdot\left[(\nabla_\mathbf{r}V(\mathbf{r}))\times\mathbf{p}\,\psi_\mathbf{k}(\mathbf{r})\right]$$

$$= \frac{i\hbar^2}{4m^2c^2}V_{\mathbf{k}'\mathbf{k}}(\mathbf{k}'\times\mathbf{k})\cdot\boldsymbol{\sigma}_{\sigma'\sigma}\,a^\dagger_{\mathbf{k}'\sigma'}a_{\mathbf{k}\sigma}$$

$$\psi_\mathbf{k}(\mathbf{r}) = e^{i\mathbf{k}\cdot\mathbf{r}}a_{\mathbf{k}\sigma}$$

$$V_{\mathbf{k}'\mathbf{k}} = \int d\mathbf{r}\,V(\mathbf{r})e^{i(\mathbf{k}-\mathbf{k}')\cdot\mathbf{r}}$$

Cf. potential scattering $\quad H_{pot} = V_{\mathbf{k}'\mathbf{k}}\,a^\dagger_{\mathbf{k}'\sigma}a_{\mathbf{k}\sigma}$

Charge current

$$\mathbf{J}_c = q_e\mathbf{v} = \frac{q_e}{m}\mathbf{p} + \frac{q_e\hbar}{4m^2c^2}\boldsymbol{\sigma}\times(\nabla_\mathbf{r}V) \Rightarrow \frac{q_e\hbar}{m}\mathbf{k}\,a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} + \frac{iq_e\hbar}{4m^2c^2}\boldsymbol{\sigma}_{\sigma'\sigma}\times(\mathbf{k}'-\mathbf{k})V_{\mathbf{k}'\mathbf{k}}a^\dagger_{\mathbf{k}'\sigma'}a_{\mathbf{k}\sigma} = \mathbf{j}_c + \delta\mathbf{j}_c$$

$$V_{\mathbf{k}'\mathbf{k}} \to \sum_i V_i$$

i: many impurity sites

Spin current

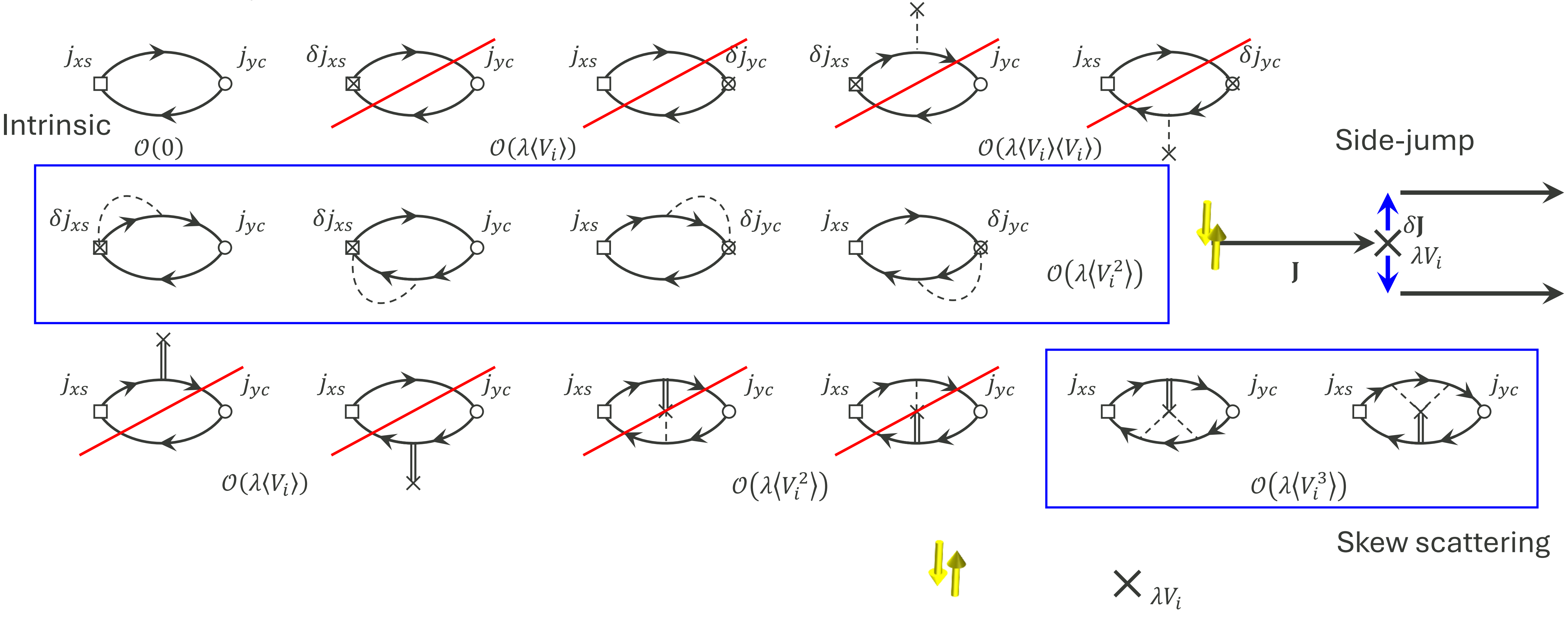
$$\mathbf{J}_s = q_e s^z\mathbf{v} = \frac{q_e\sigma^z}{2m}\mathbf{p} + \frac{q_e\hbar}{8m^2c^2}\hat{\mathbf{z}}\times(\nabla_\mathbf{r}V) \Rightarrow \frac{q_e\sigma^z\hbar}{2m}\mathbf{k}\,a^\dagger_{\mathbf{k}\sigma}a_{\mathbf{k}\sigma} + \frac{iq_e\hbar}{8m^2c^2}\hat{\mathbf{z}}\times(\mathbf{k}'-\mathbf{k})V_{\mathbf{k}'\mathbf{k}}a^\dagger_{\mathbf{k}'\sigma'}a_{\mathbf{k}\sigma} = \mathbf{j}_s + \delta\mathbf{j}_s$$

Next step: compute spin Hall conductivity in the lowest order of $\frac{\hbar^2}{m^2c^2}$

$$\sigma^{sH}_{xy}(\boldsymbol{q},\omega) = -\frac{1}{i\omega}K^{sH}_{xy}(\boldsymbol{q},\omega) \qquad K^{sH}_{xy}(\boldsymbol{q},i\Omega) = -\int_0^{\beta=1/T}d\tau\,e^{i\Omega\tau}\langle T_\tau J_{s\,\mathbf{q}x}(\tau)J_{c\,-\mathbf{q}y}(0)\rangle$$

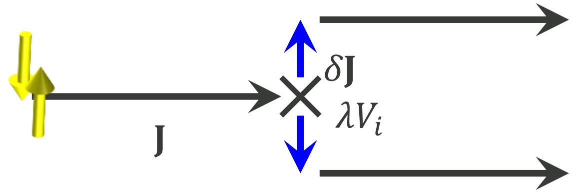
Extrinsic mechanisms for spin Hall effect and anomalous Hall effect: impurities

$$K_{xy}^{sH}(\boldsymbol{q}, i\Omega) = -\int_0^{\beta=1/T} d\tau\, e^{i\Omega\tau}\langle T_\tau\, J_{s\,\mathbf{q}x}(\tau) J_{c\,-\mathbf{q}y}(0)\rangle \qquad \lambda = \frac{\hbar^2}{m^2c^2}$$



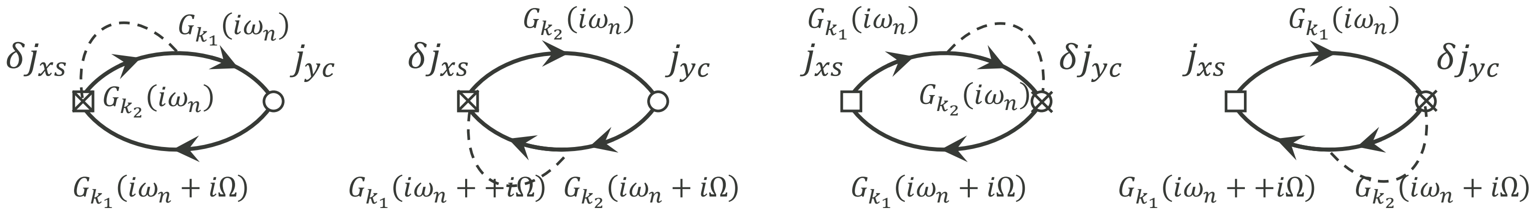
Extrinsic mechanisms for spin Hall effect and anomalous Hall effect: impurities

Side-jump

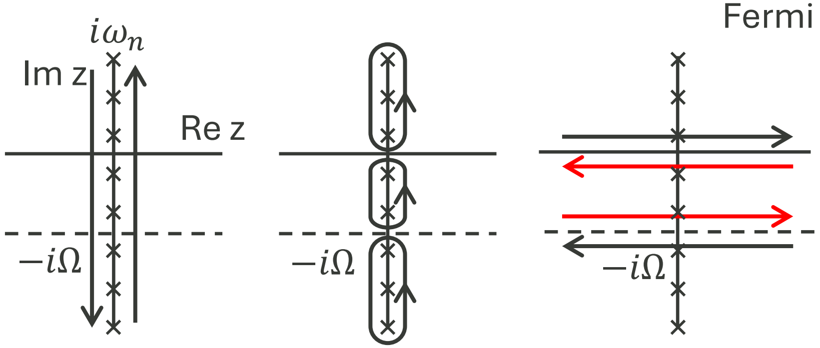


W.-K. Tse and S. Das Sarma, Phys. Rev. Lett. **96**, 056601 (2006).

Exercise: Side-jump



$$K^{sH}_{xy}(i\Omega) = i\frac{\lambda q_e^2}{8m\hbar}\sum_i V_i^2 \sum_{\mathbf{k}_1\mathbf{k}_2} k_{1y}^2\, T\sum_n \{G_{\mathbf{k}_1}(i\omega_n)G_{\mathbf{k}_1}(i\omega_n+i\Omega)G_{\mathbf{k}_2}(i\omega_n) - G_{\mathbf{k}_1}(i\omega_n)G_{\mathbf{k}_1}(i\omega_n+i\Omega)G_{\mathbf{k}_2}(i\omega_n+i\Omega)\}$$



Fermi surface contribution $= -\int \frac{dx}{2\pi i} f(x)\big[-G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^A_{k_2}(x) + G^A_{k_1}(x-\hbar\omega)G^R_{k_1}(x)G^A_{k_2}(x-\hbar\omega)$

$+G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^R_{k_2}(x+\hbar\omega) - G^A_{k_1}(x-\hbar\omega)G^R_{k_1}(x)G^R_{k_2}(x)\big]$

$$= -\int \frac{dx}{2\pi i} f(x)\big[-G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^A_{k_2}(x) + G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^R_{k_2}(x+\hbar\omega)\big]$$

$$+f(x+\hbar\omega)\big[G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^A_{k_2}(x) - G^A_{k_1}(x)G^R_{k_1}(x+\hbar\omega)G^R_{k_2}(x+\hbar\omega)\big]$$

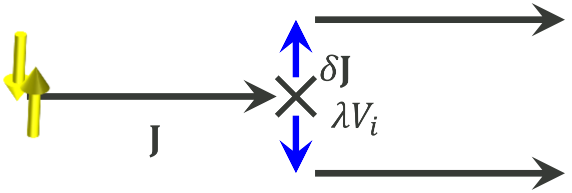
$$\to -\int \frac{dx}{2\pi i}\frac{df(x)}{dx}\hbar\omega\big[G^A_{k_1}(x)G^R_{k_1}(x)G^A_{k_2}(x) - G^A_{k_1}(x)G^R_{k_1}(x)G^R_{k_2}(x)\big]$$

$$\boxed{\sigma^{sj}_{xy} = -\frac{1}{i\omega}K^{sH}_{xy}(\hbar\omega) = i\frac{\lambda q_e^2}{8m}\sum_i V_i^2 \sum_{\mathbf{k}_1\mathbf{k}_2} k_{1y}^2 \int\frac{dx}{2\pi}\left(-\frac{df(x)}{dx}\right)G^A_{\mathbf{k}_1}(x)G^R_{\mathbf{k}_1}(x)\big[G^R_{\mathbf{k}_2}(x) - G^A_{\mathbf{k}_2}(x)\big]}$$

Extrinsic mechanisms for spin Hall effect and anomalous Hall effect: impurities

W.-K. Tse and S. Das Sarma, Phys. Rev. Lett. **96**, 056601 (2006).

Side-jump



Exercise: Side-jump

$$\sigma_{xy}^{sj} = i\frac{\lambda q_e^2}{8m}\sum_i V_i^2 \sum_{\mathbf{k}_1\mathbf{k}_2} k_{1y}^2 \int \frac{dx}{2\pi}\left(-\frac{df(x)}{dx}\right) G_{\mathbf{k}_1}^A(x) G_{\mathbf{k}_1}^R(x)\left[G_{\mathbf{k}_2}^R(x) - G_{\mathbf{k}_2}^A(x)\right]$$

$$G_{\mathbf{k}}^{R/A}(x) = \frac{1}{x - \varepsilon_{\mathbf{k}} \pm i\hbar/2\tau}$$

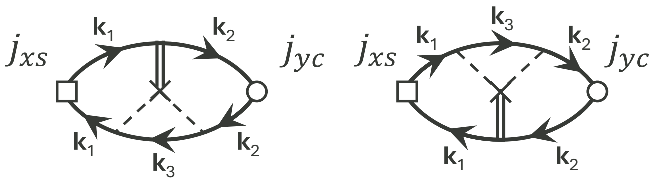
$$G_{\mathbf{k}_1}^A(x) G_{\mathbf{k}_1}^R(x) = \left(\frac{1}{x - \varepsilon_{\mathbf{k}_1} + i\hbar/2\tau} - \frac{1}{x - \varepsilon_{\mathbf{k}_1} - i\hbar/2\tau}\right)\frac{\tau}{-i\hbar} = 2\pi\delta\left(x - \varepsilon_{\mathbf{k}_1}\right)\frac{\tau}{\hbar}$$

$$G_{\mathbf{k}_2}^R(x) - G_{\mathbf{k}_2}^A(x) = \left(\frac{1}{x - \varepsilon_{\mathbf{k}_2} + i\hbar/2\tau} - \frac{1}{x - \varepsilon_{\mathbf{k}_2} - i\hbar/2\tau}\right) = -2\pi i\delta\left(x - \varepsilon_{\mathbf{k}_2}\right)$$

$$\sigma_{xy}^{sj} \propto \langle V_i^2\rangle\tau$$

$$\sigma_{xx} = \frac{q_e^2\hbar^3}{m^2}\sum_{\mathbf{k}_1} k_{1x}^2 \int \frac{dx}{2\pi}\left(-\frac{df(x)}{dx}\right) G_{\mathbf{k}_1}^R(x) G_{\mathbf{k}_1}^A(x) \propto \tau \propto 1/\langle V_i^2\rangle$$

Cf. Skew scattering



$$\sigma_{xy}^{skew} = -\frac{i\lambda q_e^2\hbar^2}{16m^2}\sum_i V_i^3 \sum_{\mathbf{k}_1\mathbf{k}_2\mathbf{k}_3} k_{1x}^2 k_{2y}^2 \int \frac{dx}{2\pi}\left(-\frac{df(x)}{dx}\right) G_{\mathbf{k}_1}^R(x) G_{\mathbf{k}_1}^A(x) G_{\mathbf{k}_2}^R(x) G_{\mathbf{k}_2}^A(x)\left(G_{\mathbf{k}_3}^R(x) - G_{\mathbf{k}_3}^A(x)\right)$$

$$\sigma_{xy}^{skew} \propto \langle V_i^3\rangle\tau^2 \sim \langle V_i^3\rangle/\langle V_i^2\rangle^2$$

Within this approach, σ_{xy}^{sj} does not scale with σ_{xx}, but σ_{xy}^{skew} scales with σ_{xx}.

More accurate analysis is developed by S. Onoda, N. Sugimoto, and N. Nagaosa (Phys. Rev. Lett. **97**, 126602 (2006)).

Time-reversal symmetry breaking

Spin Hall ⇨ Anomalous Hall

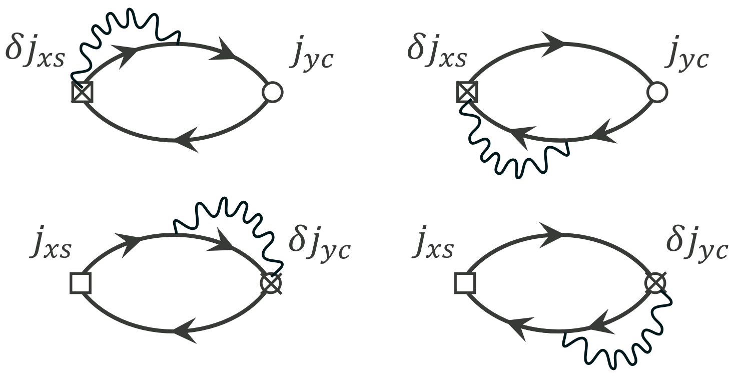
Other mechanisms for spin Hall effect: phonons

C. Gorini, U. Eckern, and R. Raimondi, Phys. Rev. Lett. **115**, 076602 (2015).
C. Xiao, Y. Liu, Z. Yuan, S. A. Yang, and Q. Niu, Phys. Rev. B **100**, 085425 (2019).

$$H_{SOC} = \frac{\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot\{(\nabla_{\mathbf{r}}V(\mathbf{r}))\times\mathbf{p}\} \Rightarrow H_{SOC} = g\frac{\hbar}{4m^2c^2}\boldsymbol{\sigma}\cdot\{(\nabla_{\mathbf{r}}\varphi(\mathbf{r}))\times\mathbf{p}\}$$

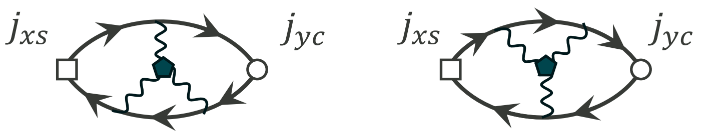
Phonon field

Side-jump



$\mathcal{O}(\lambda g^2\langle\varphi\varphi\rangle)$

Skew scattering

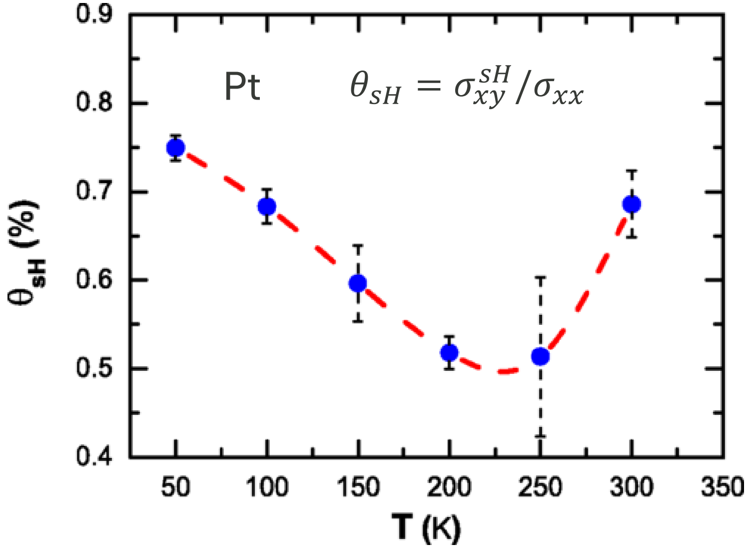


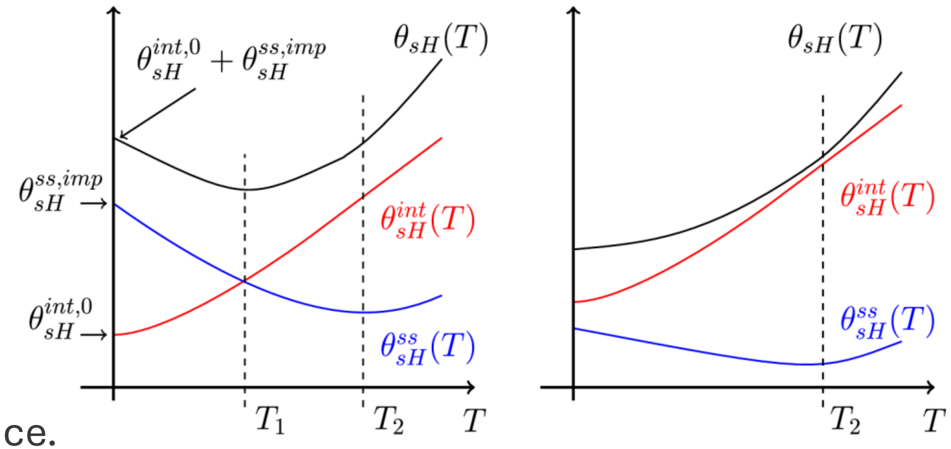
$\mathcal{O}(\lambda g^3\langle\varphi\varphi\varphi\rangle)$

This requires anharmonic term

$$H_3 = \frac{\Lambda}{3!}\int d\mathbf{r}\,\varphi^3(\mathbf{r})$$

Signature of phonon skew scattering





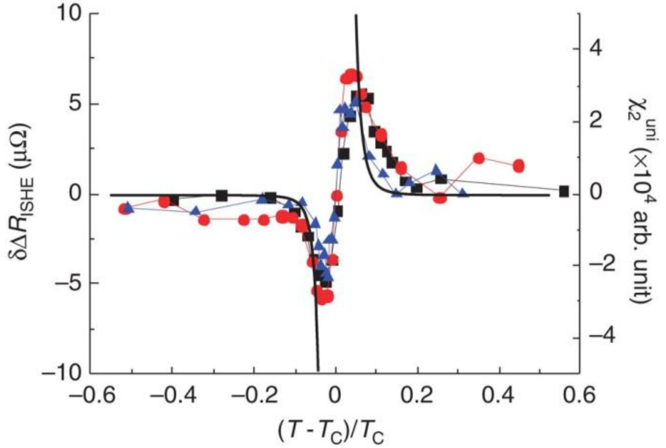
Phonon-induced spin Hall conductivity has characteristic temperature dependence.

G. V. Karnad et al. Phys. Rev. B **97**, 100405(R) (2018).

Other mechanisms for spin Hall effect: spin fluctuations

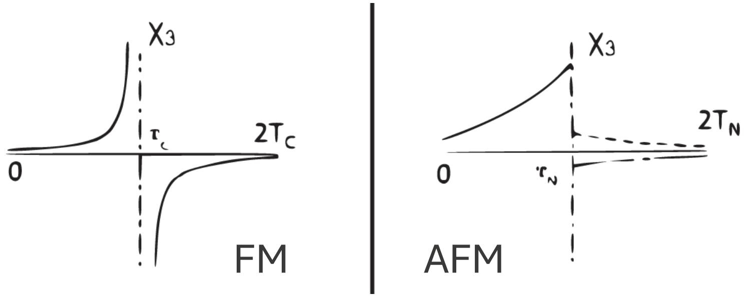
D.H. Wei, Y. Niimi, B. Gu, T. Ziman, S. Maekawa, and Y. Otani, Nat. Commun. **3**, 1058 (2012).
B. Gu, T. Ziman, and S. Maekawa, Phys. Rev. B **86**, 241303(R) (2012).

NiPd alloys



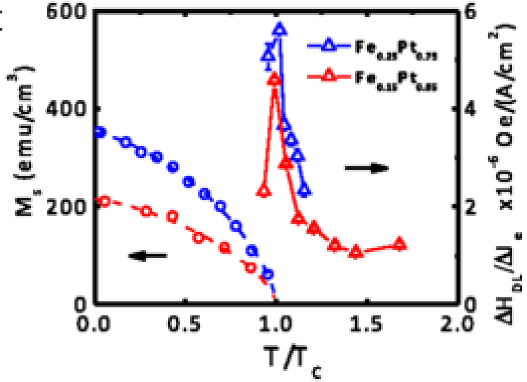
Uniform second-order nonlinear susceptibility

χ_3



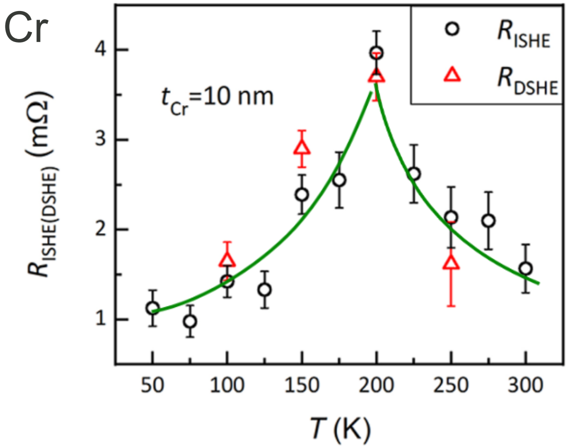
Holleis et al. npj Quantum Mater. **6**, 66 (2021).

Fe-Pt



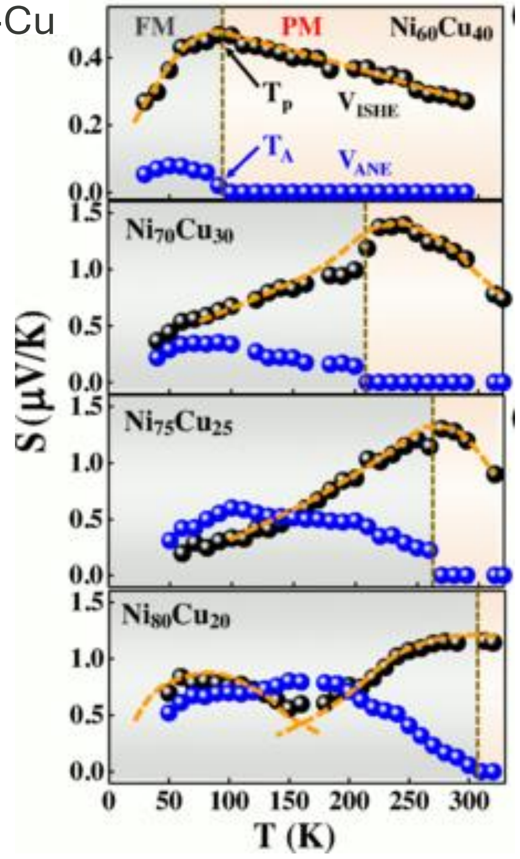
Ou et al., Phys. Rev. Lett. **120**, 097203 (2018).

Cr



Fang et al. Nano Lett. **23**, 11485 (2023).

Ni-Cu

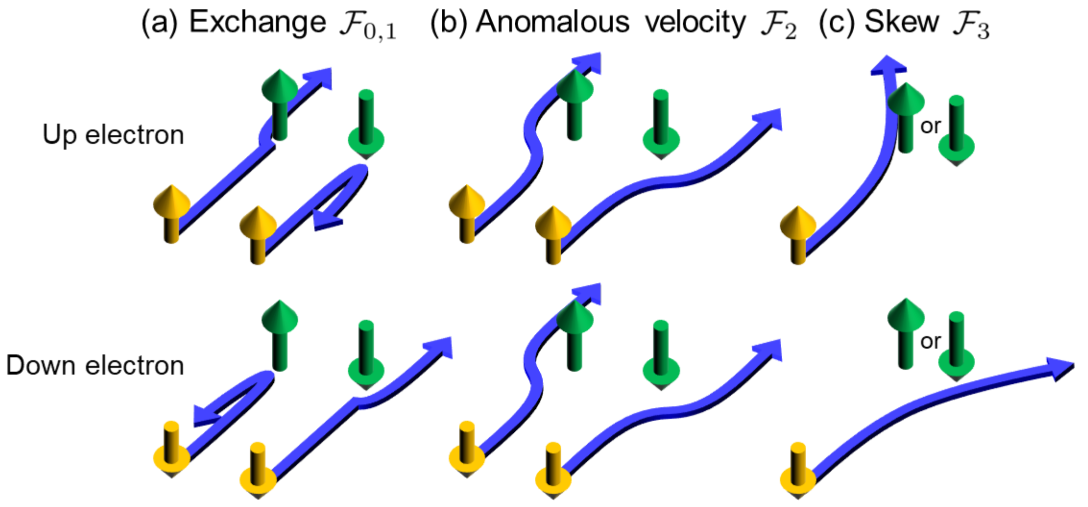


Wu et al., Phys. Rev. Lett. **128**, 227203 (2022).

New physics?

Critical spin fluctuation mechanism for spin Hall effect

S. Okamoto, T. Egami, and N. Nagaosa, Phys. Rev. Lett. **123**, 196603 (2019).
S. Okamoto and N. Nagaosa, npj Quantum Mater. **9**, 29 (2024).



s-d exchange model by J. Kondo

$$H = H_0 + H_K$$

$$H_0 = \sum_{\mathbf{k},\nu} \left(\frac{\hbar^2 k^2}{2m} - \mu \right) a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k},\nu}$$

$$H_K = -\frac{1}{\sqrt{N}} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\nu,\nu'} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}',\nu'} \Big[2(\mathbf{J}_{\mathbf{k}-\mathbf{k}'} \cdot \mathbf{s}_{\nu\nu'}) \{\mathcal{F}_0 + 2\mathcal{F}_1(\mathbf{k}\cdot\mathbf{k}')\}$$
$$+ i\mathcal{F}_2 \mathbf{J}_{\mathbf{k}-\mathbf{k}'} \cdot (\mathbf{k}' \times \mathbf{k}) + \frac{i}{\sqrt{N}} \sum_{\mathbf{p}} \mathcal{F}_3 \Big\{ (\mathbf{J}_{\mathbf{p}} \cdot \mathbf{s}_{\nu\nu'}) \left(\mathbf{J}_{\mathbf{k}-\mathbf{k}'-\mathbf{p}} \cdot (\mathbf{k}' \times \mathbf{k}) \right)$$
$$+ \left(\mathbf{J}_{\mathbf{p}} \cdot (\mathbf{k}' \times \mathbf{k}) \right) (\mathbf{J}_{\mathbf{k}-\mathbf{k}'-\mathbf{p}} \cdot \mathbf{s}_{\nu\nu'}) - \frac{2}{3} (\mathbf{J}_{\mathbf{p}} \cdot \mathbf{J}_{\mathbf{k}-\mathbf{k}'-\mathbf{p}}) (\mathbf{s}_{\nu\nu'} \cdot (\mathbf{k}' \times \mathbf{k})) \Big\} \Big]$$

J. Kondo, Prog. Theor. Phys. **27**, 772 (1962).

Velocity

Anomalous velocity

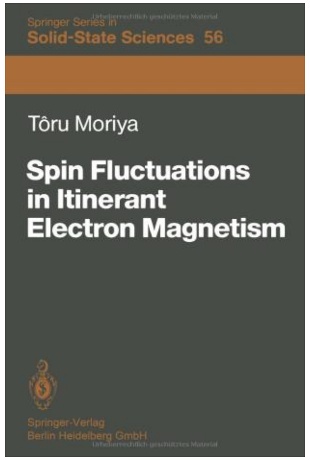
$$\mathbf{v} = \sum_{\mathbf{k}} \frac{\hbar \mathbf{k}}{m} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k},\nu} \boxed{- \frac{i\mathcal{F}_2}{\hbar\sqrt{N}} \sum_{\mathbf{k},\mathbf{k}'} \sum_{\nu,\nu'} \mathbf{J}_{\mathbf{k}-\mathbf{k}'} \times (\mathbf{k}' - \mathbf{k})\, a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k}',\nu'}}$$

$$\Rightarrow \quad \mathbf{j}^c = -e\mathbf{v}$$
$$\mathbf{j}^s = -e \left\{ \frac{1}{N} \sum_{\mathbf{k}} s^z_{\nu\nu'} a^\dagger_{\mathbf{k}\nu} a_{\mathbf{k},\nu'}, \mathbf{v} \right\}$$

This s-d exchange model contains crucial ingredients to induce spin Hall effects; skew scattering by $\mathcal{F}_3$ [Gu, Ziman, and Maekawa, Phys. Rev. B **86**, 241303(R) (2012)] and anomalous velocity by $\mathcal{F}_2$.

How can we incorporate "dynamical" spin fluctuation?

Spin fluctuations



Self-consistent renormalization (SCR) theory

Propagator of spin fluctuation

$$\frac{1}{N}\sum_{n,n'} e^{-i\mathbf{p}\cdot(\mathbf{R}_n-\mathbf{R}_{n'})}\int_0^{1/T} d\tau\, e^{i\omega_l\tau}\langle T_\tau\, \mathbf{J}_n(\tau)\,\mathbf{J}_{n'}(0)\rangle$$

$$= D_{\mathbf{p}}(i\omega_l) = \frac{1}{\delta + A|\mathbf{p}-\mathbf{Q}|^2 + |\omega_l|/\Gamma_{\mathbf{p}}}$$

Spectral function ~ inelastic neutron scattering spectra

$$-\frac{1}{\pi}\mathrm{Im}D_{\mathbf{p}}(i\omega_l \to \omega + i0_+) = \frac{1}{\pi}\frac{\omega/\Gamma_{\mathbf{p}}}{(\delta + A|\mathbf{p}-\mathbf{Q}|^2)^2 + \left(\omega/\Gamma_{\mathbf{p}}\right)^2}$$

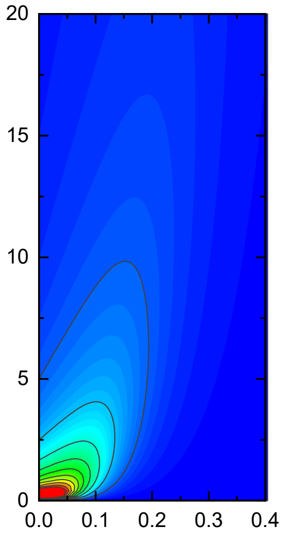
δ: distance from magnetic ordering temp.

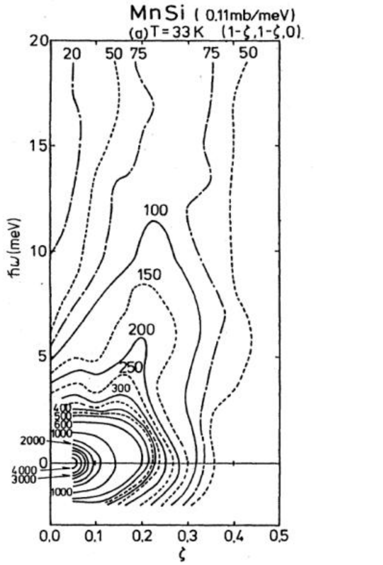
A: spin stiffness

$\mathbf{Q}$: magnetic wave vector

$\Gamma_{\mathbf{p}}$: Landau damping

Ishikawa *et al.*, Phys. Rev. B **31**, 5884 (1985)





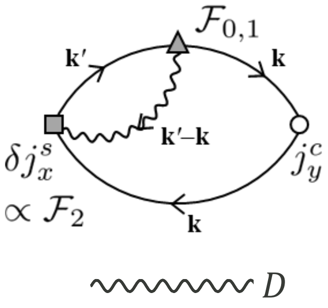
Moriya, Solid-State Sciences 56 (Springer-Verlag, Berlin, 1985).

Moriya and Kawabata, J. Phys. Soc. Jpn. **34**, 639; **35**, 669 (1973).

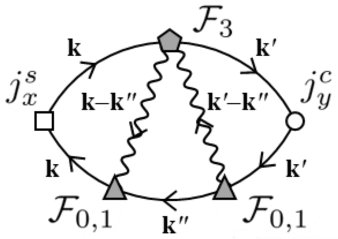
Hertz, Phys. Rev. B **14**, 1165 (1976).

Millis, Phys. Rev. B **48**, 7183 (1993).

Side jump



Skew scattering



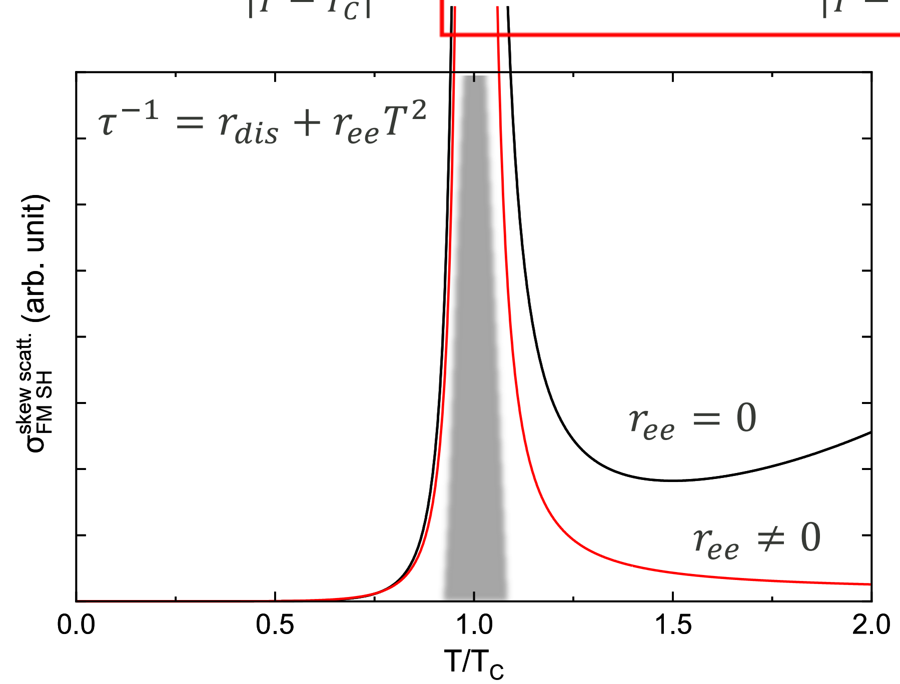
SCR theory has been successful to describe many experimental results of itinerant electron magnets, such as linear susceptibility, electrical resistivity.

This allows the examination of spin Hall effect using diagrammatic technique.

Spin Hall conductivity near the magnetic transition temperature

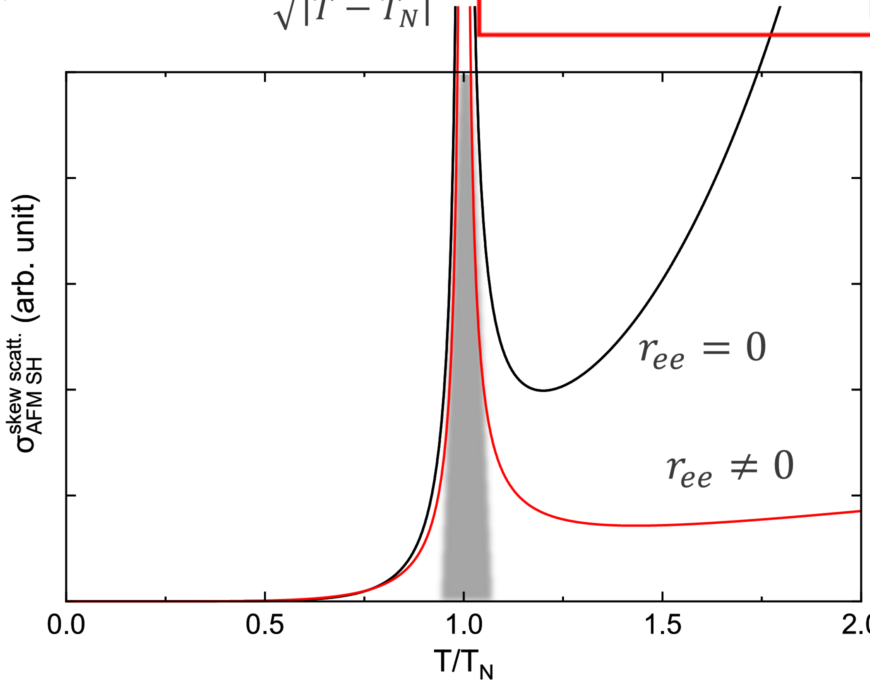
FM metal

$$\sigma_{SH,FM}^{side\ jump} \propto \mathcal{F}_0 \mathcal{F}_2\, \tau \frac{T^3}{|T - T_C|} \qquad \boxed{\sigma_{SH,FM}^{skew\ scatt.} \propto \mathcal{F}_0^2 \mathcal{F}_3\, \tau^2 \frac{T^6}{|T - T_C|^2}}$$



AFM metal

$$\sigma_{SH,AFM}^{side\ jump} \propto \mathcal{F}_0 \mathcal{F}_2\, \tau \frac{T^2}{\sqrt{|T - T_N|}} \qquad \boxed{\sigma_{SH,AFM}^{skew\ scatt.} \propto \mathcal{F}_0^2 \mathcal{F}_3\, \tau^2 \frac{T^6}{|T - T_N|}}$$



σ_{SH} by skew scattering has stronger enhancement factor $1/|T - T_{C,N}|$ than that by side jump.

Electron-electron scattering r_{ee} reduces the carrier lifetime τ and σ_{SH} at high temperatures.

FM spin fluctuation gives stronger enhancement of σ_{SH} than AFM spin fluctuation.

Spin fluctuation also influences τ and modifies the temperature dependence of σ_{SH} near $T_{C,N}$.

Quantum critical point QCP: $T_{C,N} \rightarrow 0$

Electrical resistivity of Fermi liquid (3D)

$$\rho(T) = \rho_0 + AT^2$$

Electrical resistivity at FM quantum critical point $T_C \rightarrow 0$

$$\rho(T) = \rho_0 + AT^{5/3}$$

Ueda and Moriya, J. Phys. Soc. Jpn.39, 605 (1975).

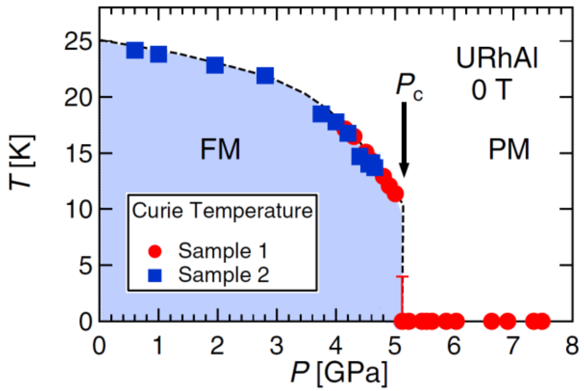
Electrical resistivity at AFM quantum critical point $T_N \rightarrow 0$

$$\rho(T) = \begin{cases} \rho_0 + AT^{3/2} & \text{Disorder dominant regime at low T} \\ \rho_0 + AT & \text{Higher T} \end{cases}$$

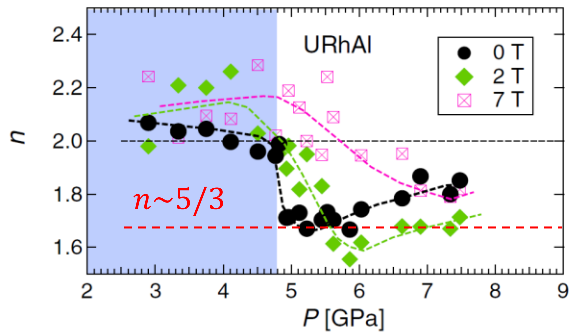
Ueda, J. Phys. Soc. Jpn.43, 1497 (1977).

Rosch, Phys. Rev. B **62**, 4945 (2000).

FM QCP in URhAl

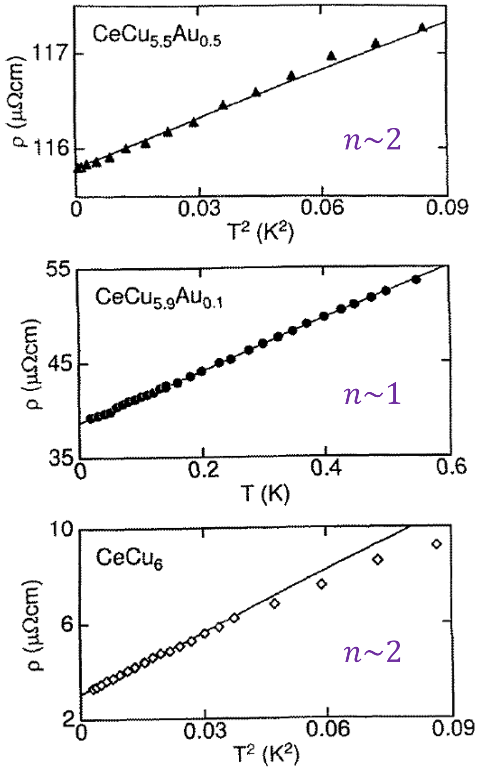


$\rho(T) = \rho_0 + AT^n$



Shimizu, Aoki, Flouquet *et al.*, Phys. Rev. B **91**, 125115 (2015).

AFM QCP in $CeCu_{6-x}Au_x$



von Löhneysen, J. Phys.: Condens. Matter **8**, 9689 (1996).

Various anomalies appear near quantum critical points, such as non-Fermi liquid like behavior of electrical resistivity.

Could quantum critical fluctuation enhance the SH effect?

Could the SH effect provide insight into the nature of QCP?

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Spin Hall conductivity near the QCP

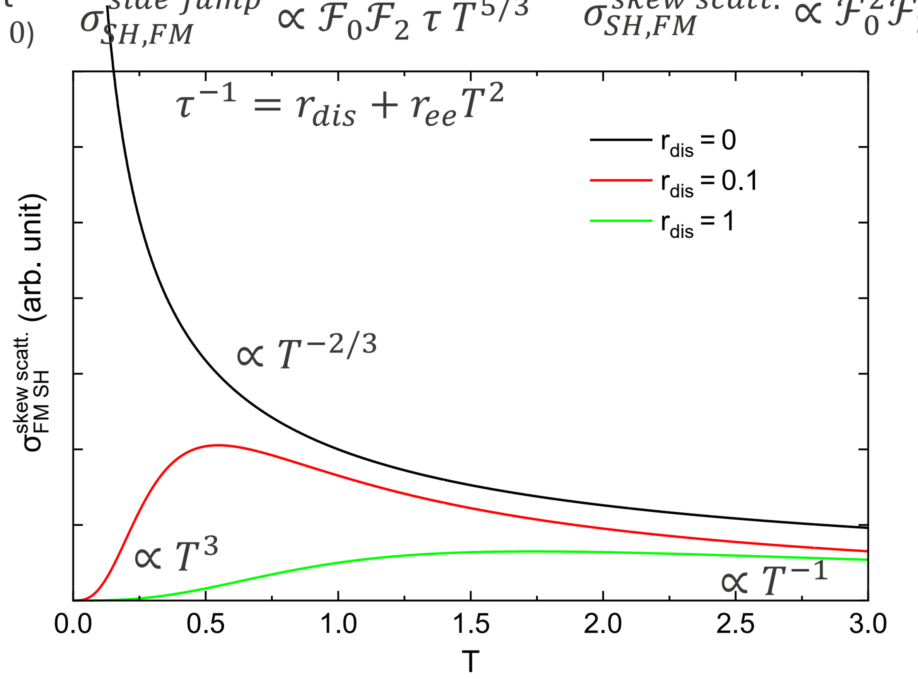
FM metal

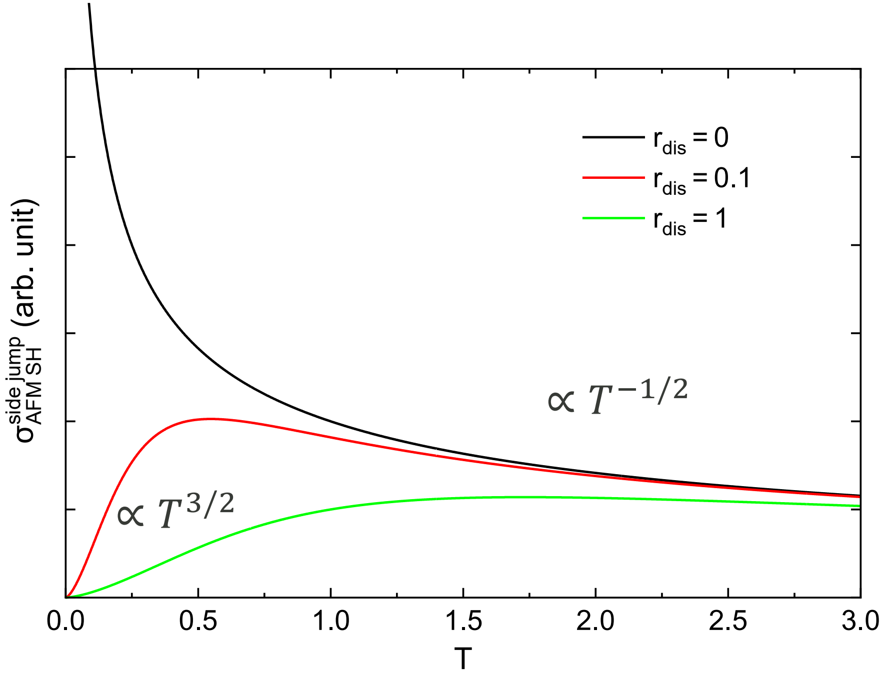
$\sigma_{SH,FM}^{side\ jump} \propto \mathcal{F}_0\mathcal{F}_2\ \tau\ T^{3/2}$ $\boxed{\sigma_{SH,FM}^{skew\ scatt.} \propto \mathcal{F}_0^2\mathcal{F}_3\ \tau^2\ T^3}$

AFM metal

$\boxed{\sigma_{SH,AFM}^{side\ jump} \propto \mathcal{F}_0\mathcal{F}_2\ \tau\ T^{3/2}}$ $\sigma_{SH,AFM}^{skew\ scatt.} \propto \mathcal{F}_0^2\mathcal{F}_3\ \tau^2\ T^{9/2}$

Clean limit
($\tau \to 0$ at $T \to 0$) $\sigma_{SH,FM}^{side\ jump} \propto \mathcal{F}_0\mathcal{F}_2\ \tau\ T^{5/3}$ $\sigma_{SH,FM}^{skew\ scatt.} \propto \mathcal{F}_0^2\mathcal{F}_3\ \tau^2\ T^{10/3}$





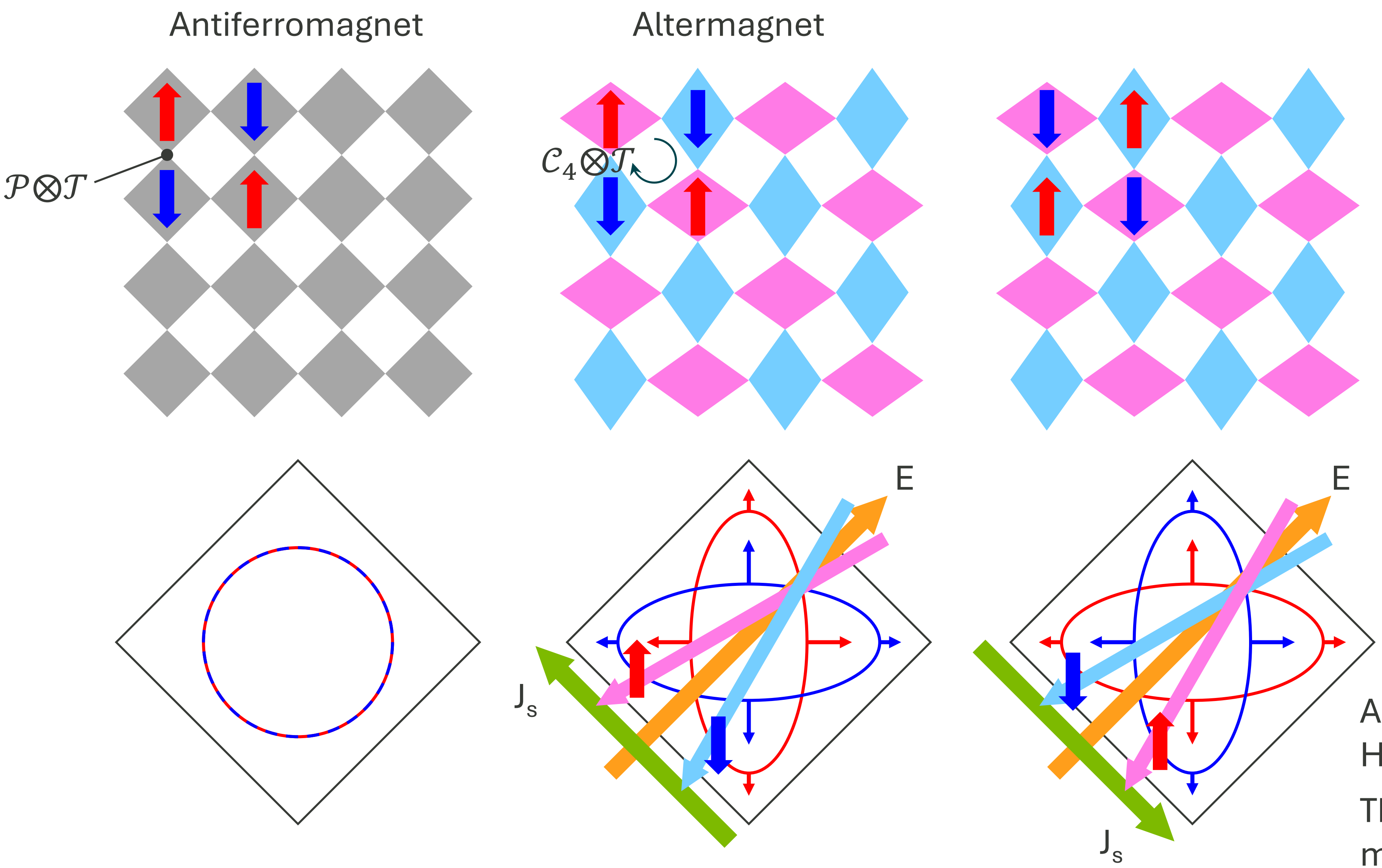
Spin fluctuation near the QCP enhances σ_{SH} at nonzero temperature in both FM and AFM.

Electron-electron scattering r_{ee} reduces the carrier lifetime τ and σ_{SH} at high temperatures.

σ_{SH} is maximized at the crossover between disorder-dominated and electron-electron scattering dominated regimes.

σ_{SH} is sensitive to the microscopic mechanism, side jump or skew scattering, and, for FM case, disorder.

Spin Hall effect without SOC: Altermagnets



Altermagnets can generate spin Hall effect without SOC.

The effect is sensitive to magnetic and structural domain.

Other applications

Chern insulators

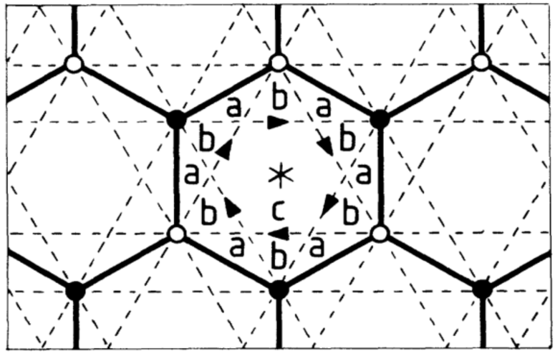
Oak Ridge National Laboratory

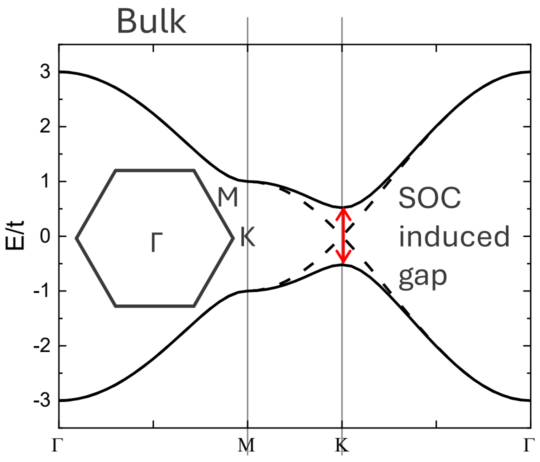
Quantum Hall effect to Chern insulator

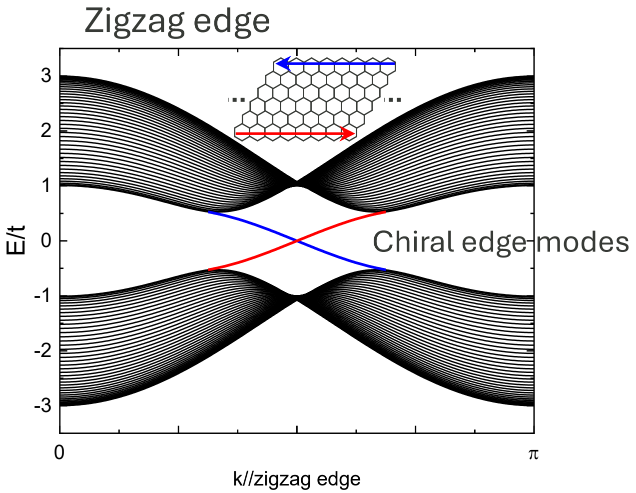
Landau levels are not necessary.
We need just filled topological bands with nonzero Chern number.

F. D. M. Haldane, Phys. Rev. Lett. **61**, 2015 (1988).

Theoretical prediction

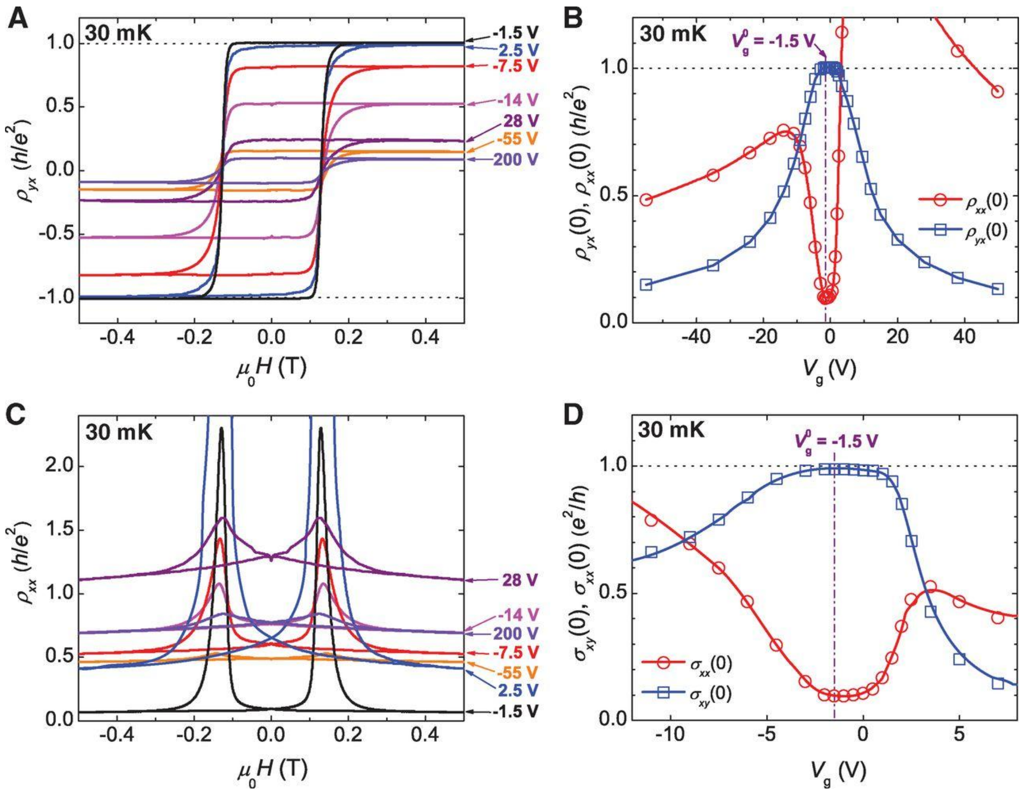






C.-Z. Chang et al. Science **340**, 167 (2013).

Experimental realization using Cr-doped $(Bi,Sb)_2Te_3$



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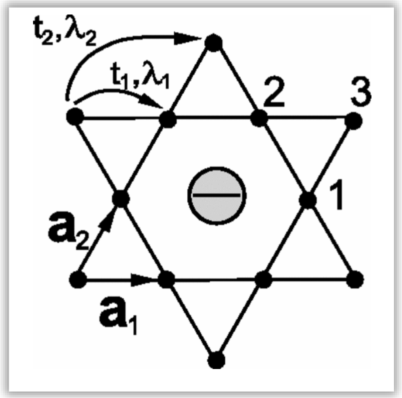
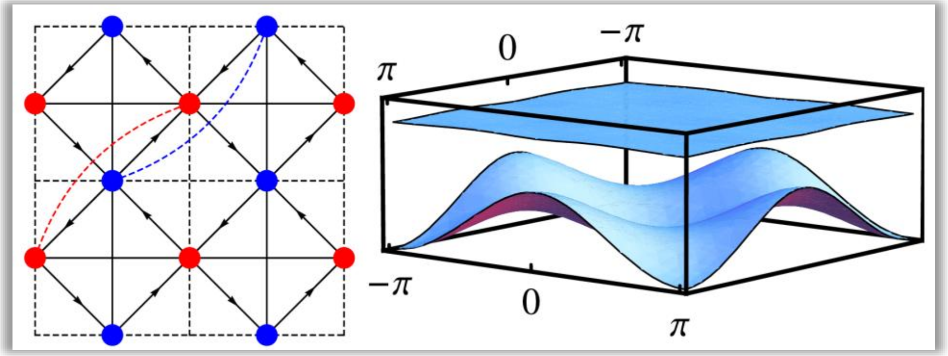
Signature of Chern number $|\mathcal{C}| = 1$ and quantized Hall conductivity $\sigma_{xy} = e^2/h$

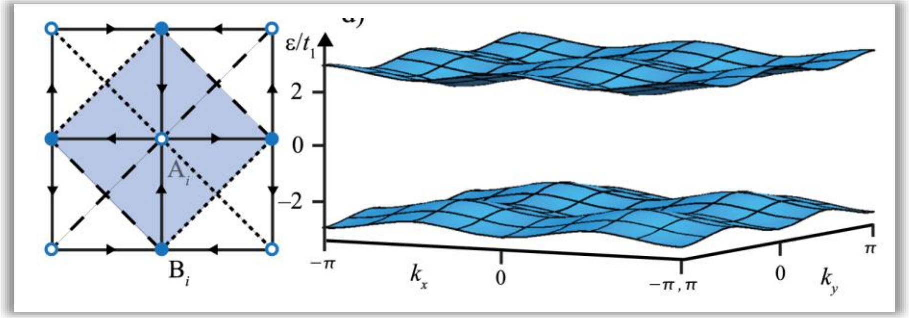
Fractional quantum Hall effect to fractional Chern insulator

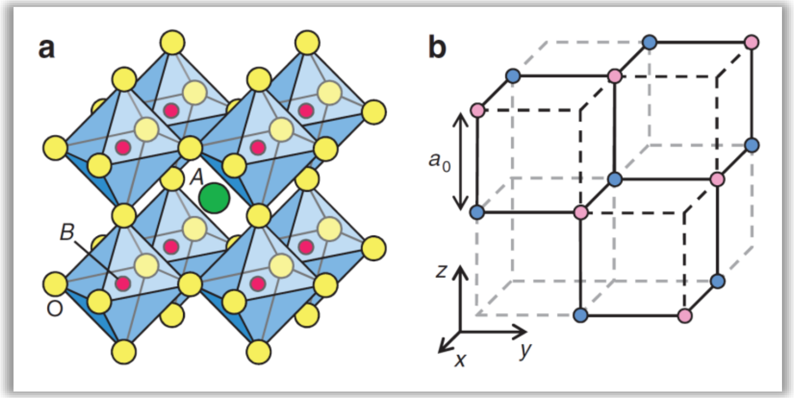
Landau levels are not necessary.
We need just flat bands with nonzero Chern number, fractional filling, and repulsive interactions.

E. Tang, J.-W. Mei, and X.-G. Wen, Phys. Rev. Lett. **106**, 236802 (2011). kagome
K. Sun, Z. Gu, H. Katsura, S. Das Sarma, Phys. Rev. Lett. **106**, 236803 (2011). checkerboard
T. Neupert, L. Santos, C. Chamon, and C. Mudry, Phys. Rev. Lett. **106**, 236804 (2011). π flux
D.N. Sheng, Z.-C. Gu, K. Sun, and L. Sheng, Nat. Commun. **2**, 389 (2011).
D. Xiao, W. Zhu, Y. Ran, N. Nagaosa, and S. Okamoto, Nat. Commun. **2**, 596 (2011). Transition-metal oxide (111) bilayers



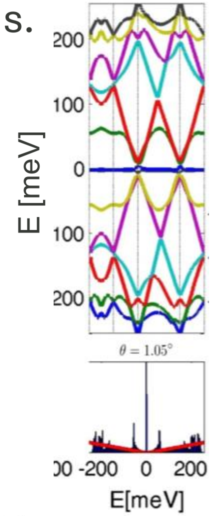


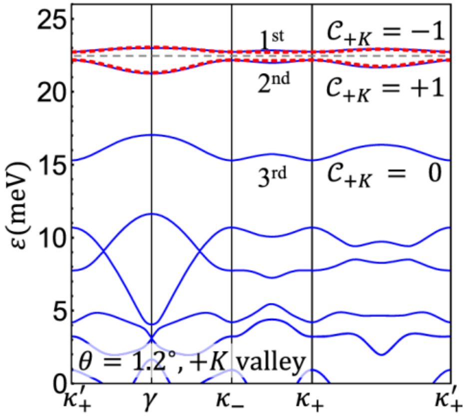




Fractional quantum Hall effect to fractional Chern insulator

Landau levels are not necessary.
We need just flat bands with nonzero Chern number, fractional filling, and repulsive interactions.





Theoretical prediction of flat bands

R. Bistritzer and A. H. MacDonald, PNAS **108** (30) 12233 (2011). twist bilayer graphene

F. Wu, A. MacDonald, et al. Phys. Rev. Lett. **122**, 086402 (2019). twist bilayer $MoTe_2$

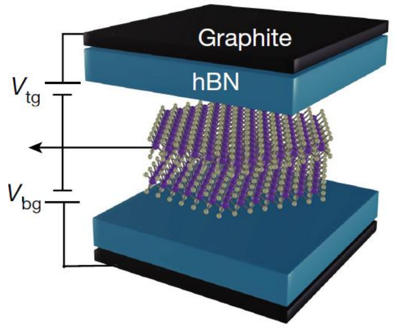
TBG

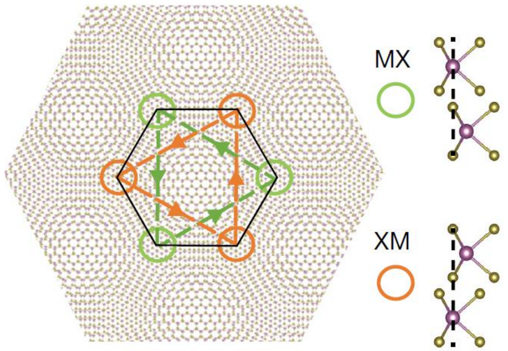
Y. Xie, A. Vishwanath, P. Jarillo-Herrero, A. Yacoby et al. Nature **600**, 439 (2021).

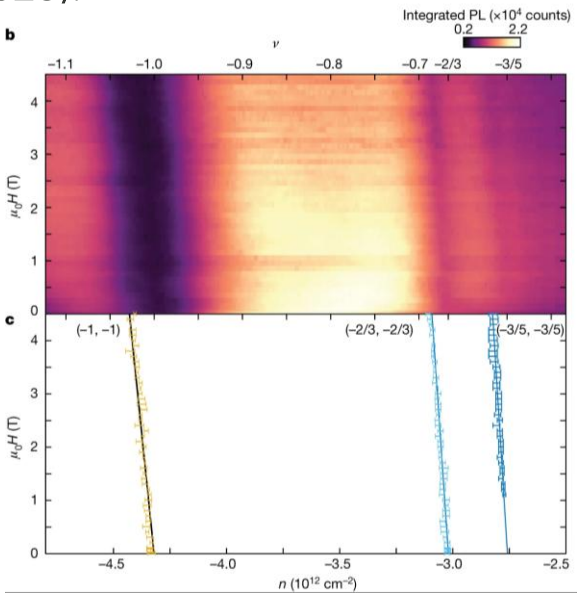
Twist bilayer of $MoTe_2$

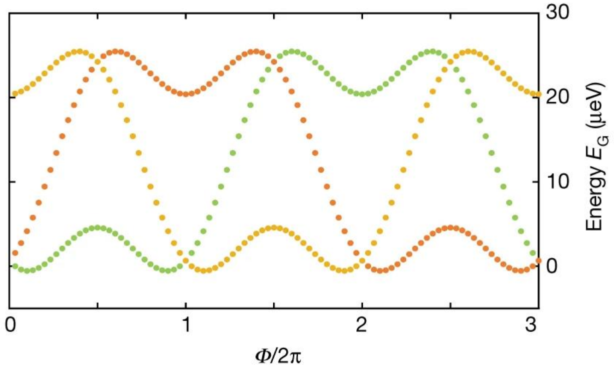
J. Cai, Y. Ran, L. Fu, D. Xiao, W. Yao, X. Xu et al. Nature **622**, 63 (2023).

Y. Zeng, K. F. Mak, J. Shan et al. Nature **622**, 69 (2023).









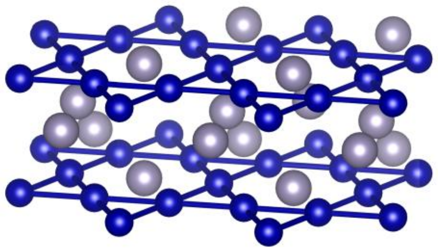
Numerical diagonalization using a finite size cluster with twist boundary condition

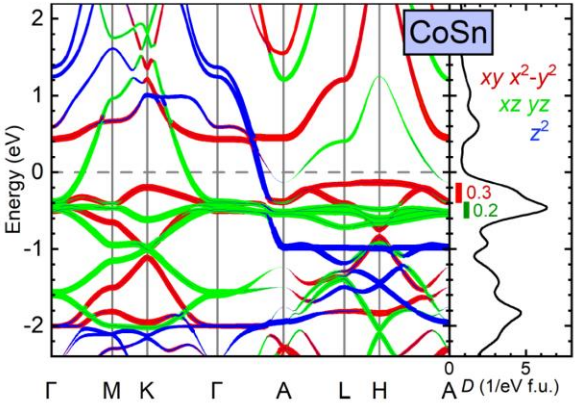
Concept by Q. Niu et al. Phys. Rev. B **31**, 3372 (1985).

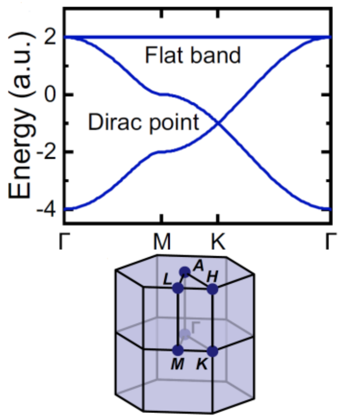
Theory and computation played important roles.

Fractional Chern insulator in a multiorbital Kagome system

CoSn-type compounds

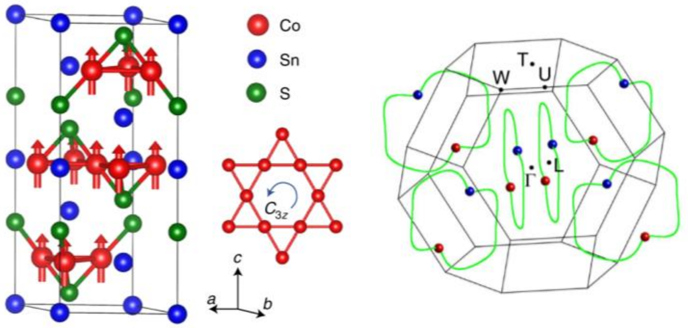






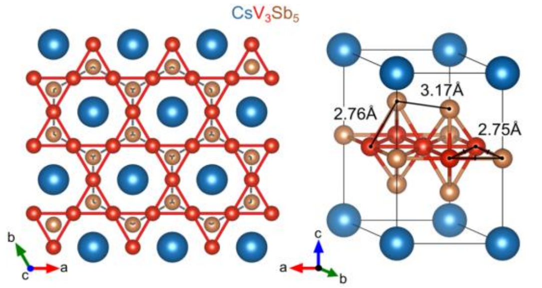
Meier, S.O. et al., Phys. Rev. B **102**, 075148 (2020).

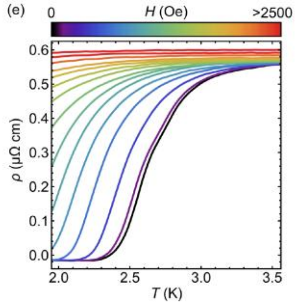
$Co_3Sn_2S_2$ Liu et al., Nat. Phys. **14**, 1125 (2018).



Weyl fermions

AV_3Sb_5



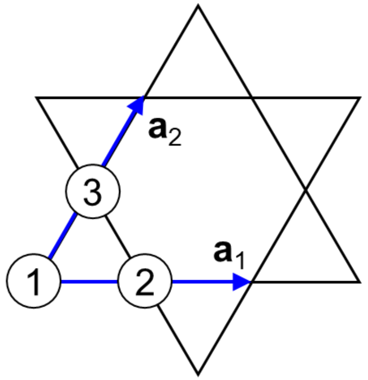


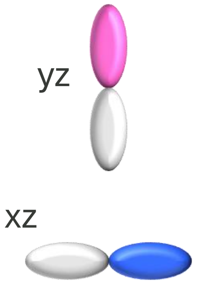
Ortiz et al., Phys. Rev. Materials **3**, 094407 (2019);
Phys. Rev. Lett. **125**, 247002 (2020).

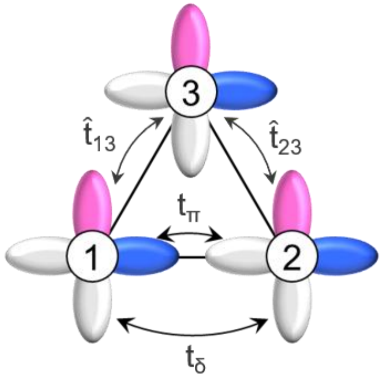
Superconductivity
Nontrivial topology

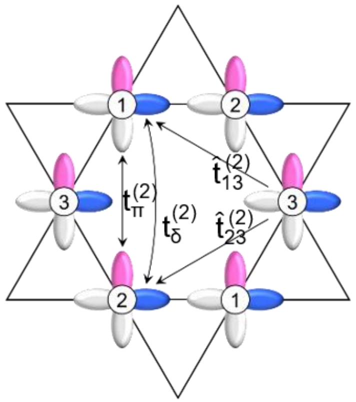
Orbitals induce a lot of complexity and variety of phenomena.

This work Two-orbital model on a Kagome lattice mainly for CoSn-type intermetallic compounds









Hopping $H_t = \sum_{\langle \mathbf{r}\,\mathbf{r}'\rangle} \sum_{\alpha\beta\sigma} \left(t_{\mathbf{r}\mathbf{r}'}^{\alpha\beta} c_{\mathbf{r}\alpha\sigma}^\dagger c_{\mathbf{r}'\beta\sigma} + \mathrm{H.c.} \right)$

Slater-Koster parametrization

“Atomic” spin-orbit coupling

$$H_{SOC} = \lambda \vec{l} \cdot \vec{s} = \frac{\lambda}{2} \sum_\sigma \left[i\sigma_{\sigma\sigma}^z c_{yz,\sigma}^\dagger c_{xz,\sigma} + \mathrm{H.c.} \right]$$

Topological property and correlation effects

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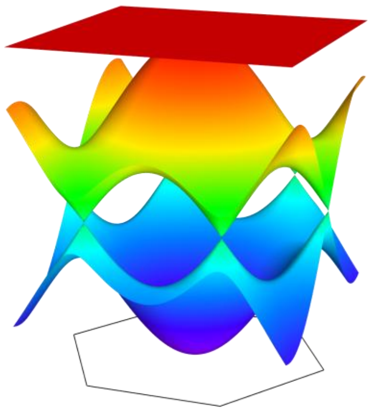
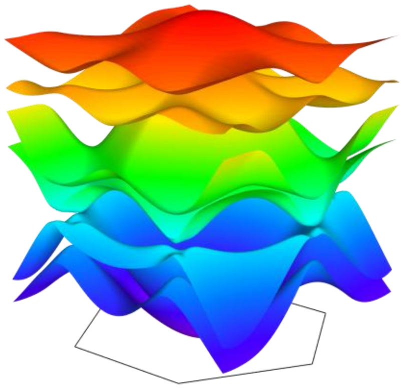
Fractional Chern insulator in a multiorbital Kagome system

Single-particle properties

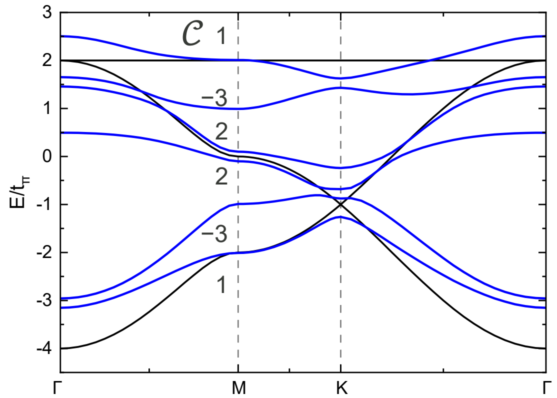
$t_\delta = t_\pi$, $\lambda = 0$



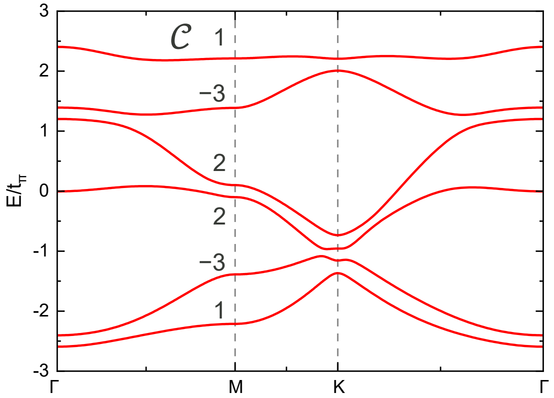
$t_\delta = 0.5\, t_\pi$, $\lambda = 0.2\, t_\pi$



Only nearest-neighbor hopping

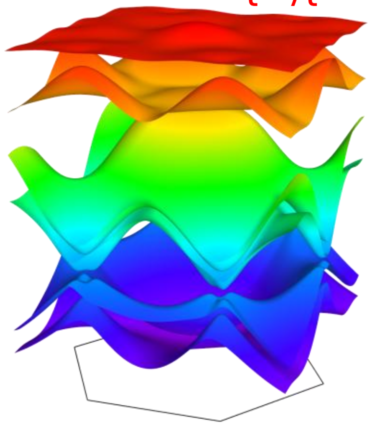


Additional second-neighbor hopping



$t_\delta = 0.5\, t_\pi$, $\lambda = 0.2\, t_\pi$

$t^{(2)}/t^{(1)} = -0.2$



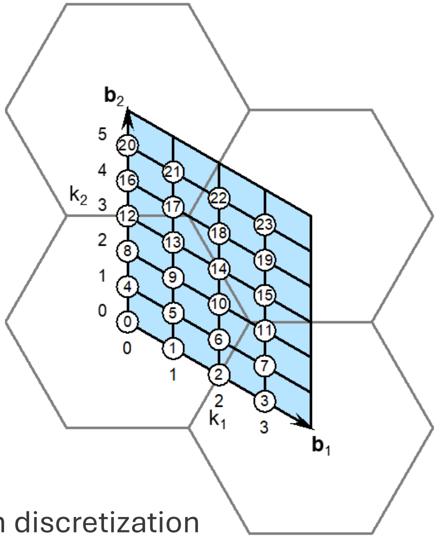
Many-body effects

Numerical diagonalization of an effective model

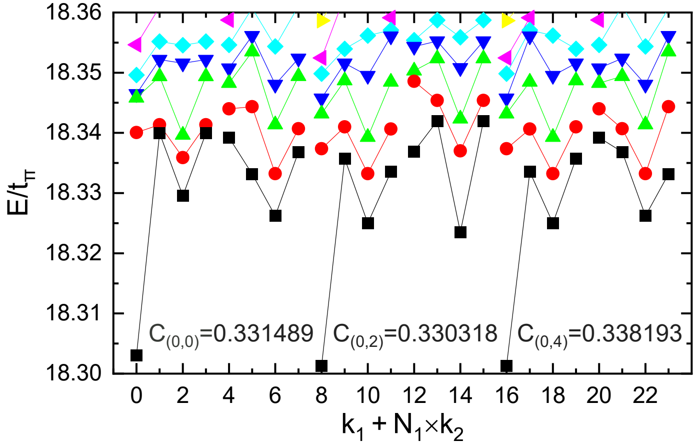
$$H_{eff} = \sum_{\mathbf{k}} \varepsilon_{1\mathbf{k}}\, \psi^\dagger_{1\mathbf{k}} \psi_{1\mathbf{k}} + H_{U,eff}$$

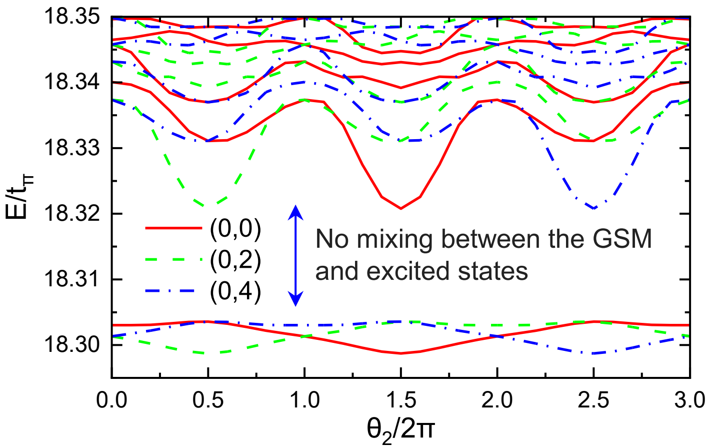
1/3 filling of the top flat band

U'−J=2, V=1



Momentum discretization





Spectral flow of the ground state manifold upon flux insertion (twist boundary)

Kagome systems could have topological flat bands, potentially leading to even more exotic fractional Chern insulator.

S. Okamoto, N. Mohanta, E. Dagotto, and D. N. Sheng, Commun. Phys. **5**, 198 (2022).

Oak Ridge National Laboratory

Summary

Various Hall effects

- Ordinary Hall effect
- Integer quantum Hall effect
- Fractional quantum Hall effect
- Anomalous Hall effect
- Spin Hall effect

Ordinary Hall effect

- Semiclassical description
- Example of material specifics FeTe

Anomalous Hall effect

- Berry curvature in momentum space
- Berry curvature in real space:
  topological Hall effect
- Example

FeTe, transition-metal oxide heterostructures

Spin Hall effect

- Intrinsic mechanisms
- Extrinsic mechanisms
- Phonon mechanisms
- Spin fluctuation mechanisms
  Critical spin fluctuation
- Altermagnets

Chern insulators

- Integer Chern insulators
- Fractional Chern insulators
  Multiorbital Kagome system

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